Part II:
Defining Lines on Surfaces

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Line Drawings from 3D Models
SIGGRAPH 2005 Course Notes

Types of Lines

Lines can convey:

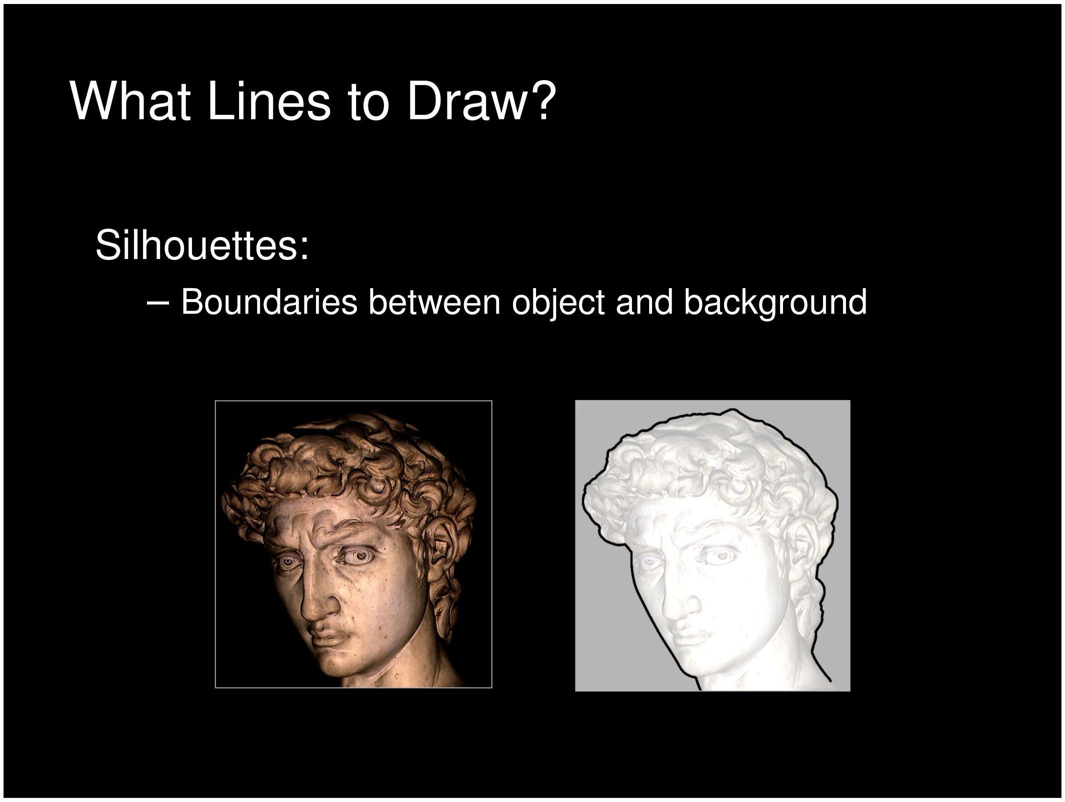
- Tone (lighting, shadows, etc.)
- Material (color, roughness, texture, etc.)
- Discontinuities (part boundaries, etc.)
- Shape

Lines can convey many different things, including various combinations of lighting, material, surface markings, discontinuities, and shape. Part of what makes good artists good, in fact, is knowing how to select lines that simultaneously convey several of these cues. For the purposes of this course, though, we will limit ourselves to lines that are intended to convey shape.

Lines for Conveying Shape

[Flaxman 1805]

Here's a hand-drawn illustration by John Flaxman that illustrated a 19th century translation of the Odyssey. Notice how there are a variety of lines illustrating various effects, including things like shading, but in particular there are many lines that convey shape. When a viewer looks at these lines, these lines are naturally interpreted as indicating shape: they are not perceived as lines drawn on the surface!

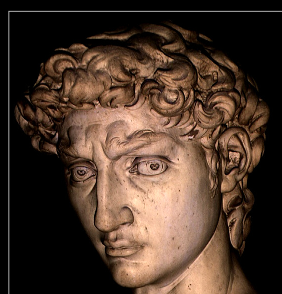


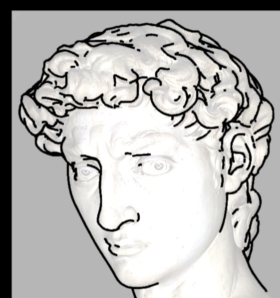
So, let’s say we want to do this for a 3D object, such as Michelangelo’s David. We can ask the question of what kinds of lines artists draw. First, we start with the silhouette: the boundary between the object and the background. Here, we draw the silhouette on the right, superimposed over a contrast-reduced version of the photograph on the left (just so we can see what’s going on). Silhouettes are obviously very important, and are an essential ingredient in any line drawing. However, as you can see here, they’re clearly not enough.

What Lines to Draw?

Occluding contours:

- Depth discontinuities
- Surface normal perpendicular to view direction



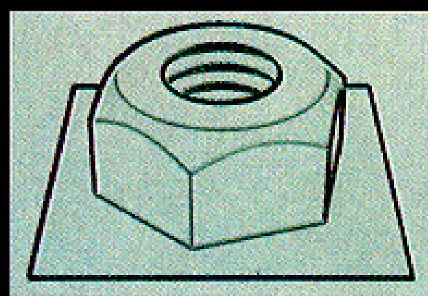


[Saito & Takahashi 90, Winkenbach & Salesin 94, Markosian et al 97, …]

Another thing commonly included is a generalization of silhouettes called occluding contours. These mark any depth discontinuities, not just those against the background. As seen on the right, this adds a lot of important detail to the drawing, but still does not convey shallow features, particularly those viewed head-on. Still, these are a common component in NPR systems.

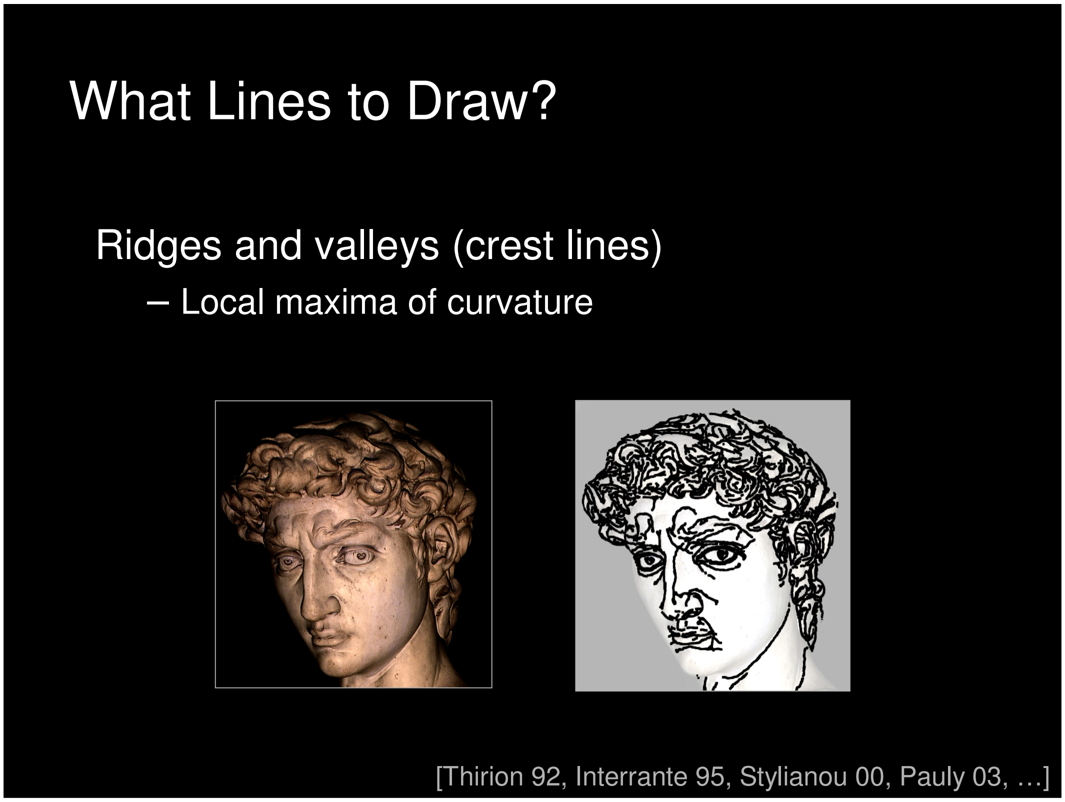
What Lines to Draw?

Creases: infinitely sharp folds



[Saito & Takahashi 90]

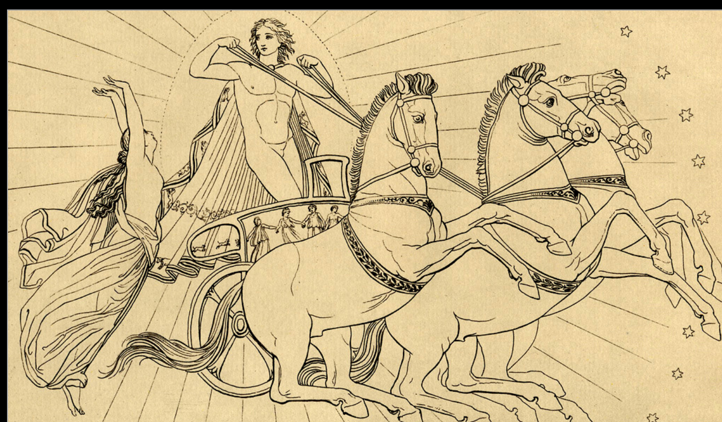
How can we add more detail? One approach that has been proposed is to draw sharp creases of the surface. This works well for polyhedral objects, and is a frequent ingredient in technical drawings. The algorithm is very simple: just look for a dihedral angle (that is, the angle between two faces connected along an edge) smaller than a threshold.



Unfortunately, the natural generalization for smooth surfaces, ridge and valley lines, leads to mixed results. If you look at the picture on the right, you can see that some of the ridge and valley lines, such as those around the eyes, do a good job of marking features. However, other lines look like surface markings and no good artist would include them in a hand-made drawing.

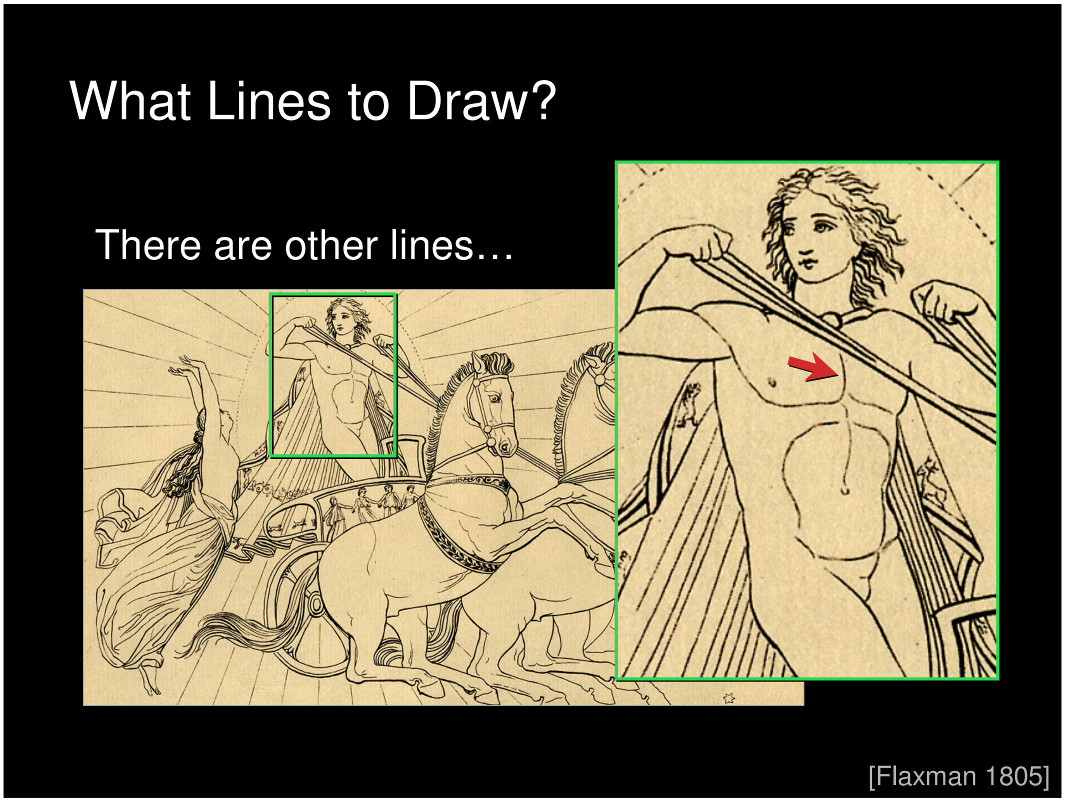
What Lines to Draw?

There are other lines…

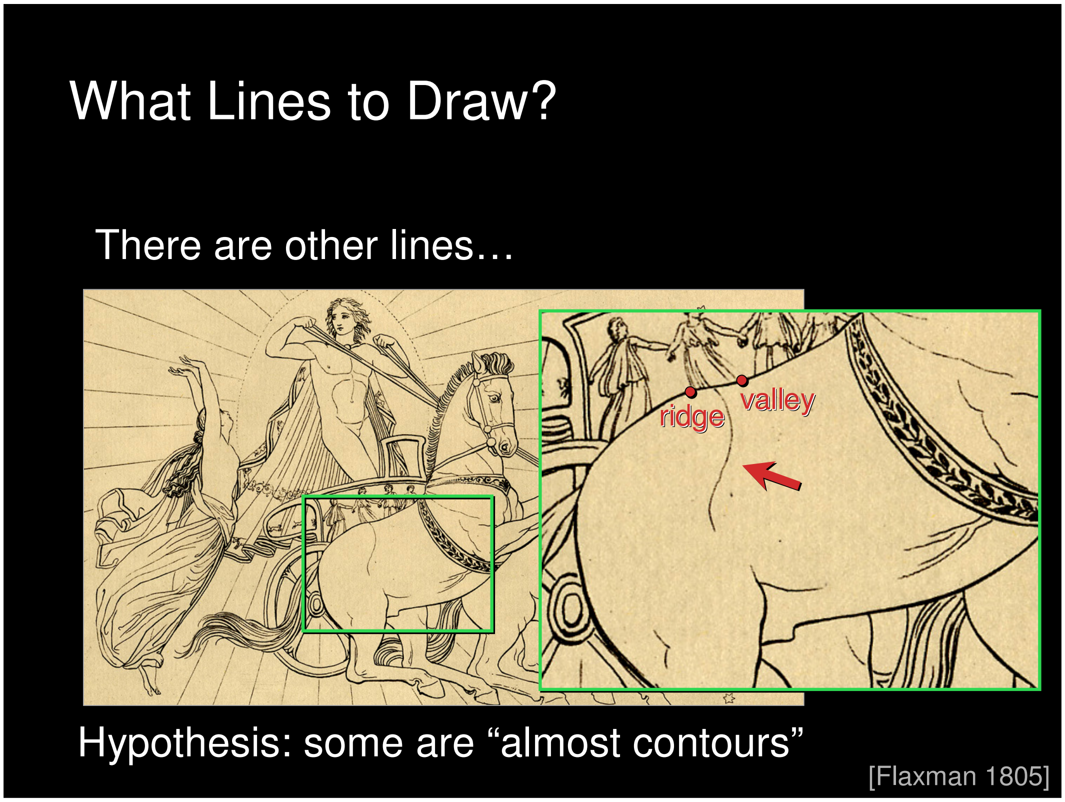


[Flaxman 1805]

So, we're still left with the problem of coming up with another family of lines. If we go back and look at a real line drawing, we see that there are, in fact, more lines that pretty clearly are not contours, ridges, or valleys.



For example, there’s this line on the chest that’s clearly on the right side of the torso, not in the middle of the “valley”.



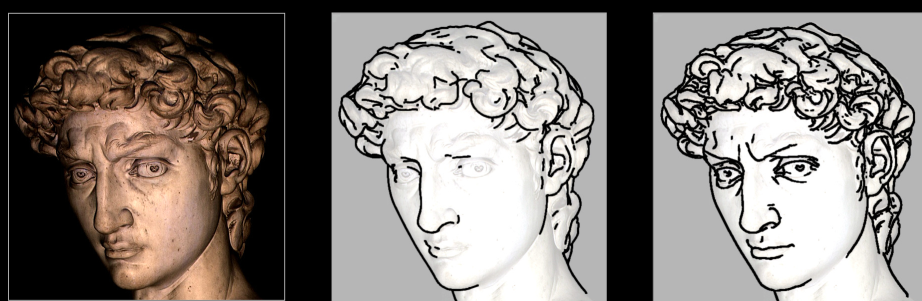
Another example is the line on the back of the horse. See how it misses the ridges and valleys on the back of the horse. Also, it’s certainly not a contour.

We hypothesize that these lines are “almost contours”: if you moved your head a bit to the left, this line would in fact become a contour. We call these lines “suggestive contours”, and we’ll later see how to formalize what they are and how to find them on a surface.

Suggestive Contours

"Almost contours":

- Points that become contours in nearby views



contours

contours + suggestive contours

So, here's what suggestive contours look like on the David. You can see that they complement the contours nicely (in fact, you can prove that they line up with the contours in the image). Also, they include a lot of the detail that's missing in the contours-only drawing.

Differential Geometry

Many shape-conveying lines based on surface differentials

First-order: surface normals

- Example: occluding contours occur where normal perpendicular to view direction

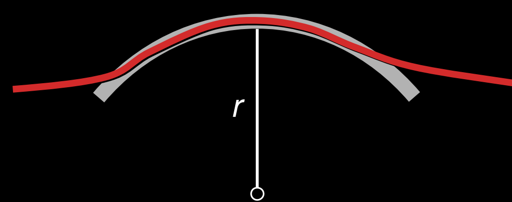
Other lines need higher-order derivatives

It turns out that in order to formalize different families of 3D lines on a surface, we'll need some math from a field called differential geometry. This is the field that concerns itself with what it means to take "derivatives" of curves and surfaces. You are already familiar with a first-order differential quantity of surfaces: the normal. In fact, as has already been mentioned, occluding contours critically depend on the normal: they are zeros of the dot product between the normal and the view direction. In a very similar way, different kinds of lines, like suggestive contours, will have definitions that depend on higher-order derivatives.

Differential Geometry

Many lines based on curvatures

- Second-order differential properties of surface
- For a curve: reciprocal of radius of circle that best approximates it locally

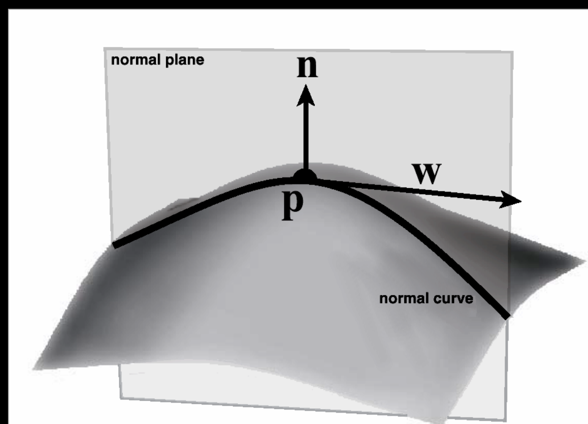


- For a surface: ?

So, let's move on to exploring second-order derivatives, or curvatures. To start with, let's recall the familiar definition of the curvature of a curve: at each point, it is the reciprocal of the radius of a circle that best approximates the curve locally. The sharper the bend in the curve, the higher the curvature. Curvature has units of one-over-length: if you scale an entire curve up by a factor of two, all the curvatures are halved.

Normal Curvature

Curvature of a normal curve



For a surface, we can talk about the curves formed by intersecting the surface with any plane containing the normal. These are called “normal curves”, and their curvature is “normal curvature”. So, for each point on the surface, there are many different curvatures, corresponding to all the different normal planes passing through that point.

Curvature on a Surface

Normal curvature varies with direction, but for a smooth surface satisfies

$$\kappa_n = \begin{pmatrix} s & t \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

$$= \begin{pmatrix} s & t \end{pmatrix} \mathbf{II} \begin{pmatrix} s \\ t \end{pmatrix}$$

for a direction (s,t) in the tangent plane and a symmetric matrix $\mathbf{II}$

There is something interesting that happens, though. For a smooth surface, the variation of normal curvature with direction can't be arbitrary – it has a very specific form. Imagine setting up a local orthonormal coordinate system in the tangent plane at a point on a surface. For any direction (s,t), expressed in terms of that coordinate system, we can find the normal curvature in that direction in terms of a simple formula involving a symmetric matrix II. This matrix is known as the "second fundamental tensor", and as we'll see is related to how much the surface is bent. Note that if you were to expand this formula you'd get terms quadratic in s and t: this whole expression is therefore just a fancy way of writing a quadratic form.

Interpretation of $\mathbf{II}$

Second-order Taylor-series expansion:

$$z(x, y) = \tfrac{1}{2} e x^2 + f x y + \tfrac{1}{2} g y^2$$

"Hessian": second partial derivatives

$$\mathbf{II} = -\begin{pmatrix} \mathbf{s}_{uu} \cdot \mathbf{n} & \mathbf{s}_{uv} \cdot \mathbf{n} \\ \mathbf{s}_{uv} \cdot \mathbf{n} & \mathbf{s}_{vv} \cdot \mathbf{n} \end{pmatrix}$$

Derivatives of normal

$$\mathbf{II} = \begin{pmatrix} \mathbf{n}_u \cdot \hat{\mathbf{u}} & \mathbf{n}_u \cdot \hat{\mathbf{v}} \\ \mathbf{n}_v \cdot \hat{\mathbf{u}} & \mathbf{n}_v \cdot \hat{\mathbf{v}} \end{pmatrix}$$

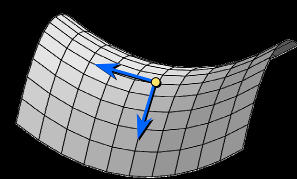
What exactly is II? What are its elements? It turns out that there are many formulas for them, all involving some notion of the second derivative of the surface. For example, they are precisely the second-order terms in a Taylor series expansion of the surface (assuming that z is oriented along the surface normal). Equivalently, they are the derivatives of the normal as you move along the surface.

Incidentally, if you go looking for formulas like this in various textbooks, there's a good chance you may see them written with the opposite sign. This is because when writing the formulas you need to establish the conventions of whether normals are considered to point into or out of the surface, and in addition whether convex surfaces are taken to have positive or negative curvature. We assume that convex surfaces have positive curvature, and we use the usual graphics convention of outward-pointing normals, leading to the signs used here.

Principal Curvatures and Directions

Can always rotate coordinate system so that **II** is diagonal:

$$\mathbf{II} = \mathbf{R}^{\mathrm{T}} \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \mathbf{R}$$



κ_1 and κ_2 are *principal curvatures*, and are minimum and maximum of normal curvature

Associated directions are *principal directions*

Eigenvalues and eigenvectors of **II**

In many situations it is convenient to rotate the local coordinate system to make the matrix II diagonal. It turns out you can always do this: the new coordinate axes are the eigenvectors of II. (You might recall a neat theorem from linear algebra that the eigenvalues of symmetric matrices are guaranteed to be real: here's a real-life application that relies on this fact.)

Once you've done this change of coordinates, the new axes are known as the principal directions, and the corresponding curvatures are the principal curvatures. If we plug in the new form of II into the formula for normal curvature, we see that all normal curvatures have to lie between the principal curvatures. So, the principal curvatures are the minimum and maximum curvatures for any direction (at that point on the surface).

Gaussian and Mean Curvature

The Gaussian curvature $K = \kappa_1 \kappa_2$

The mean curvature $H = ½ (\kappa_1 + \kappa_2)$

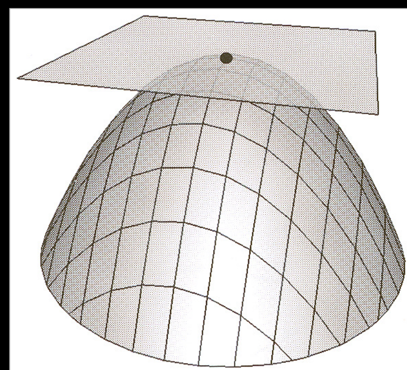
Equal to the determinant and half the trace, respectively, of the curvature matrix

Enable qualitative classification of surfaces

When talking about curvatures, there are a couple more terms that often crop up: Gaussian curvature and mean curvature. These are equal to the product and average of the principal curvatures, and can also be computed directly from II (expressed in terms of any coordinate system), as the determinant and trace. Notice one interesting feature about Gaussian curvature: it has units of "curvature squared", which is different from all the other flavors of curvature we've talked about.

Gaussian and mean curvature are very useful for qualitatively talking about the shape of a surface.

Positive Gaussian Curvature: Elliptic Points

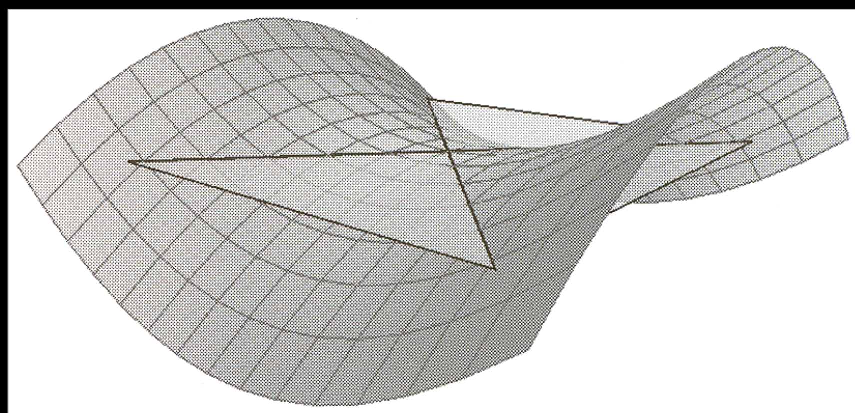


Convex/concave depending on sign of H

Tangent plane intersects surface at 1 point

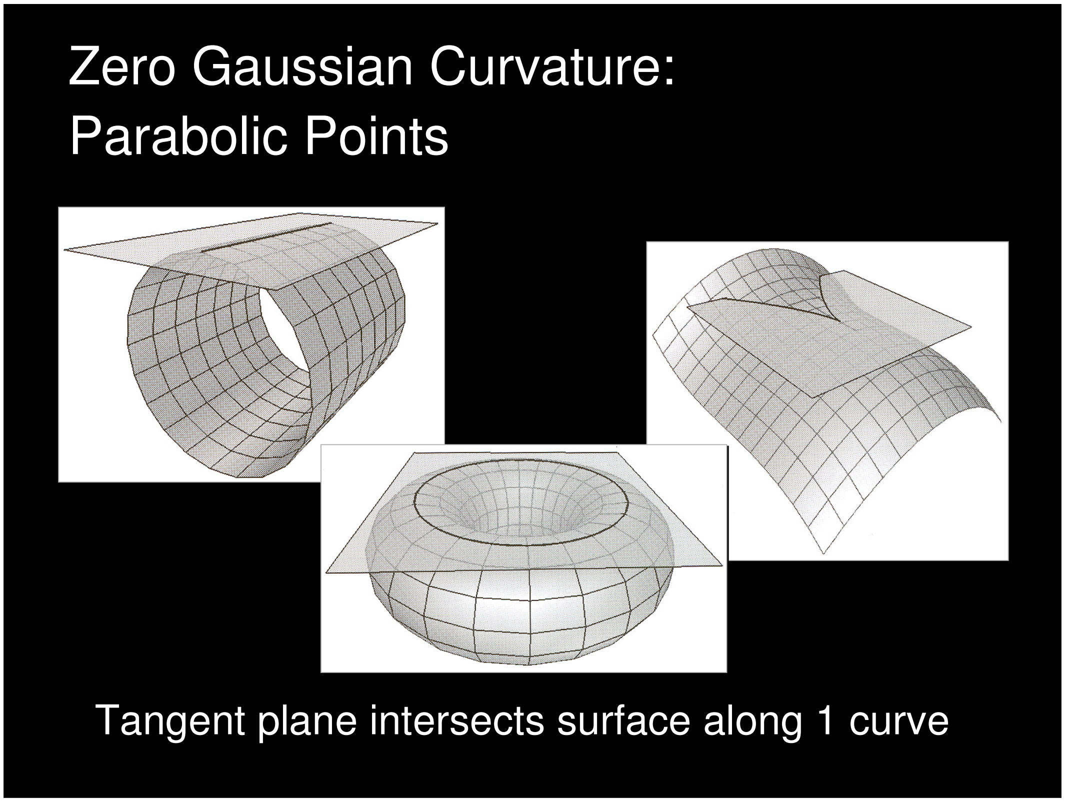
The most basic classification looks at the sign of Gaussian curvature. If it is positive, then the principal curvatures are either both positive or both negative (and you can tell which one by looking at the sign of the mean curvature). Points of positive Gaussian curvature are known as elliptic points, and are either convex or concave regions.

Negative Gaussian Curvature: Hyperbolic Points



Tangent plane intersects surface along 2 curves

If the Gaussian curvature is negative, we have what are known as hyperbolic points, at which the surface is saddle-shaped. So, if you look in one direction the surface is convex, while in the perpendicular direction the surface is concave.



Finally, we have parabolic points, at which one of the principal curvatures is zero. The most basic shape with zero Gaussian curvature is a cylinder, but there are many more complex surfaces at which K=0 as well. In general, except for degenerate cases like cylinders, the parabolic points will form curves on the surface (known as parabolic lines), separating regions of positive and negative Gaussian curvature.

Historical Note

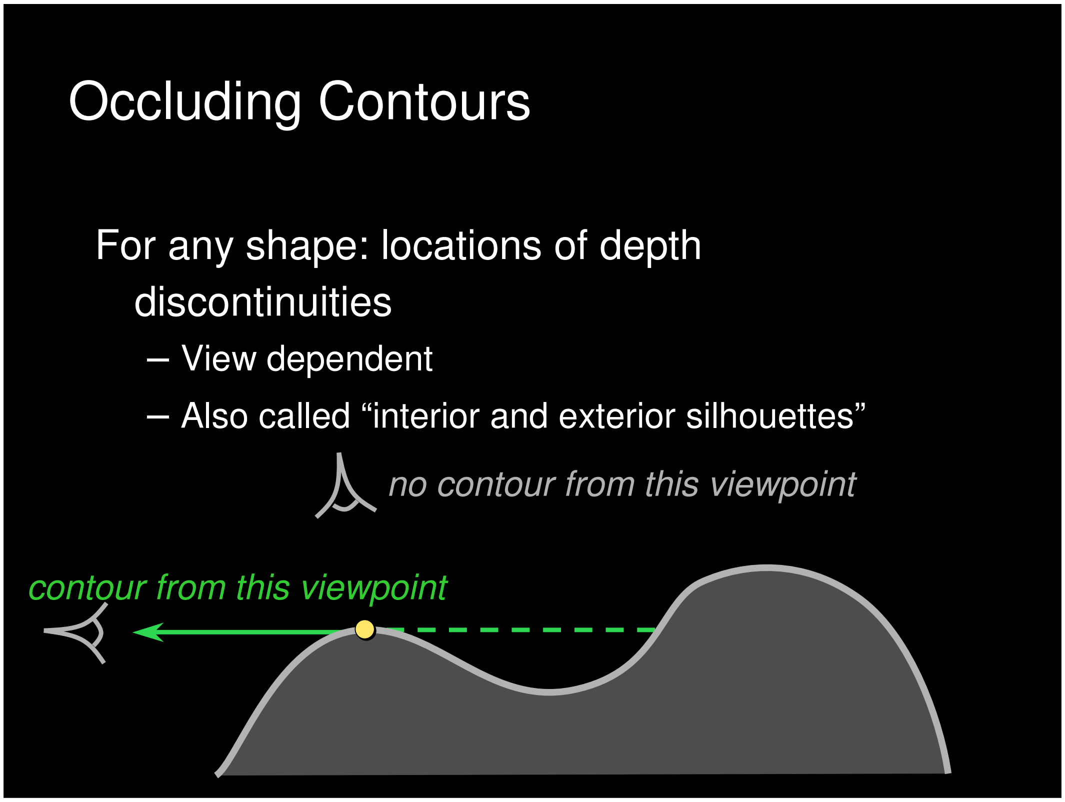
Mathematician Felix Klein was convinced that parabolic lines held the secret to a shape's aesthetics, and had them drawn on the Apollo of Belvedere…

He soon abandoned the idea...

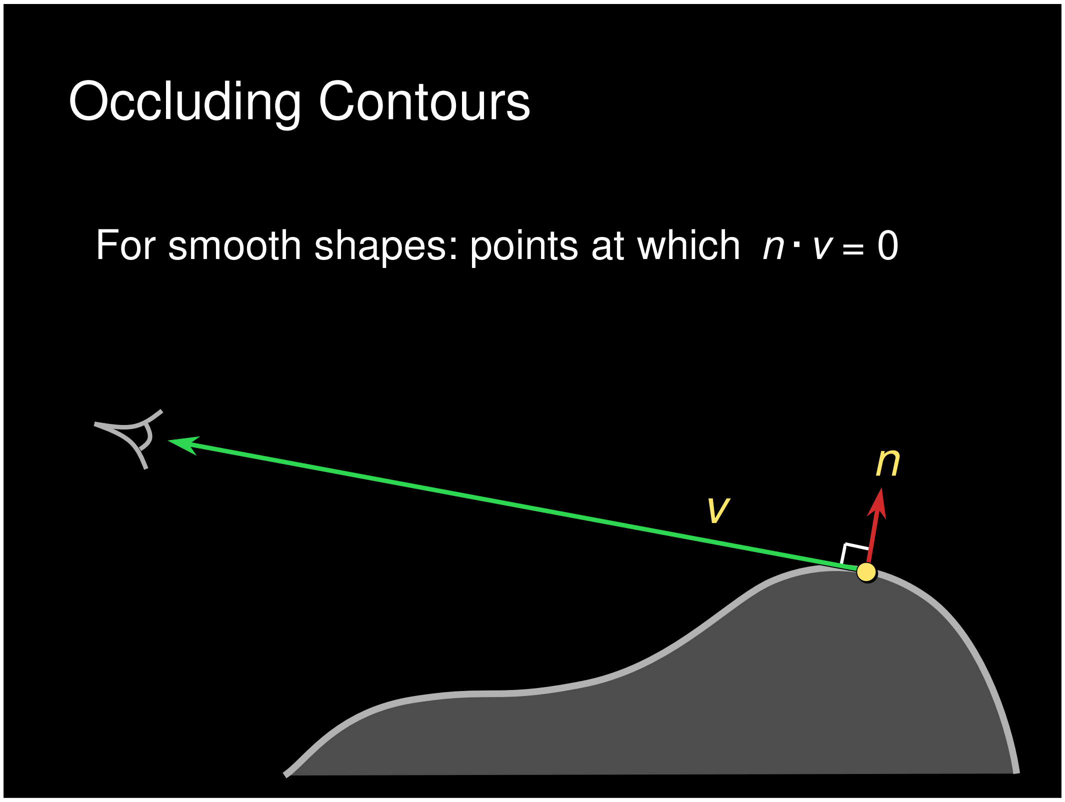
[Hilbert & Cohn-Vossen]

As long as we're talking about parabolic lines in a course about line drawings, it would be remiss not to relate an anecdote about the mathematician Felix Klein, who thought that parabolic lines might, in fact, be interesting lines to draw on a surface. He had them drawn (probably by a poor grad student) on the Apollo of Belvedere. Unfortunately, the experiment didn't turn out that great.

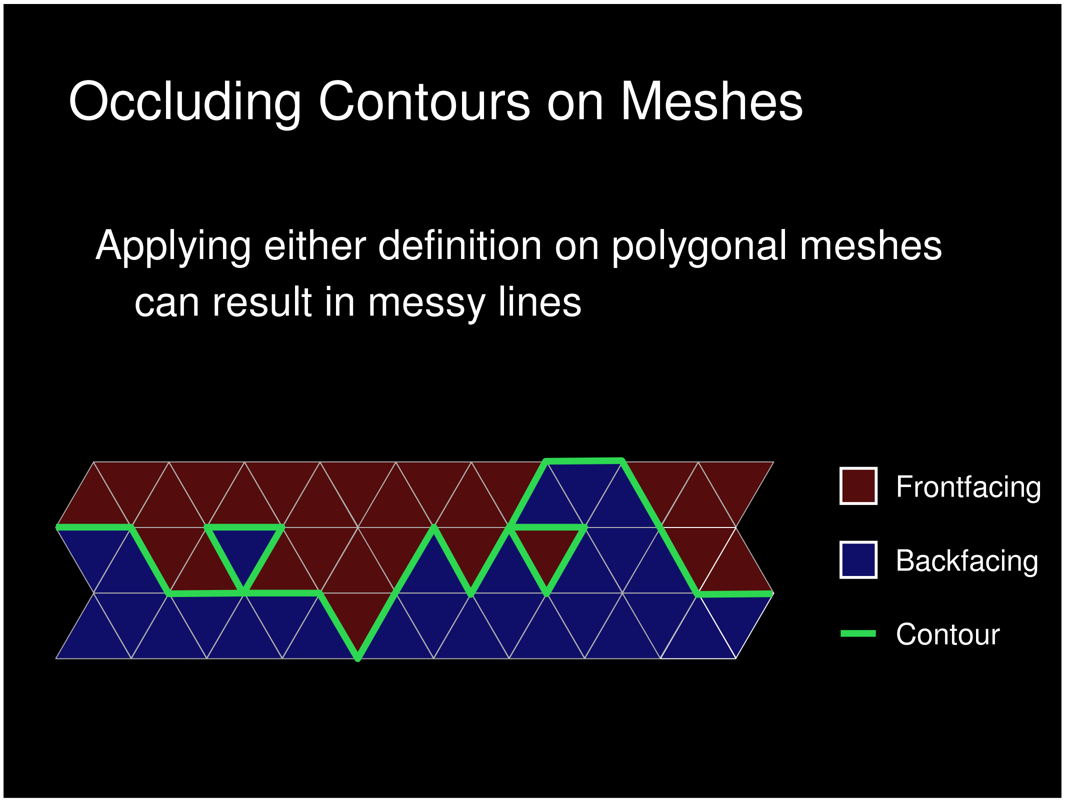
A little later, we'll see that Klein wasn't entirely wrong: it is possible to select a subset of the parabolic lines that look OK…



Now that we have some math under our belts, let us look in more detail at the different types of lines. We begin with occluding contours, sometimes also called "interior and exterior silhouettes." There are a few different ways of defining these, of which a very straightforward definition is simply those locations at which, from the current viewpoint, there is a depth discontinuity. Note that these are view-dependent lines, which implies both benefits and drawbacks. On the plus side, the view dependence makes it much more likely that these lines are interpreted as conveying shape, rather than as surface markings. On the other hand, this means that the lines will have to be recomputed for each frame, and potentially makes it harder to do things like line drawings in stereo.



On smooth surfaces, there is another definition of contours that is useful: contours are those surface locations where the surface normal n is perpendicular to the viewing direction v. That is, places where n dot v is equal to zero.

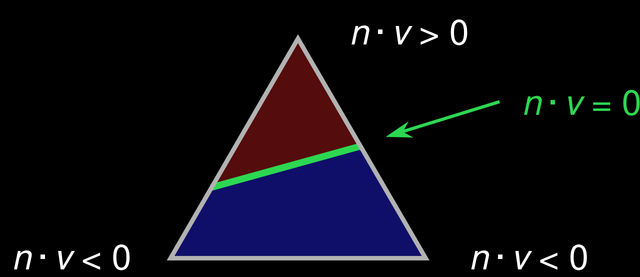


We can apply the same principles to polyhedra (i.e., polygonal meshes), but there's a problem. With only a tiny bit of noise, we can run into a situation where polygons along the boundary are "just barely" front- or back-facing, and the boundary between them is not a simple curve: it can entirely surround certain faces. This isn't necessarily a problem if the only thing you're doing is drawing the curve, since it will be viewed edge-on. However, if you are doing any further processing on the curve, such as trying to draw them with stylization, this can lead to big problems.

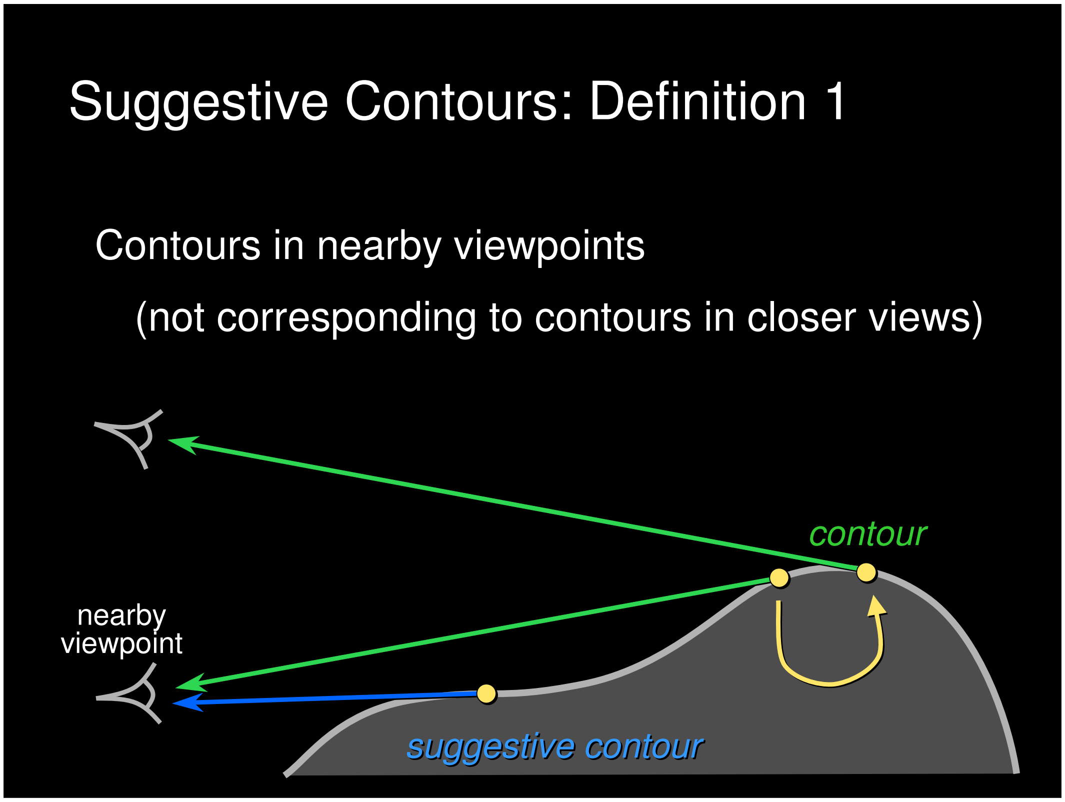
Occluding Contours on Meshes

Alternative: interpolate normals within faces

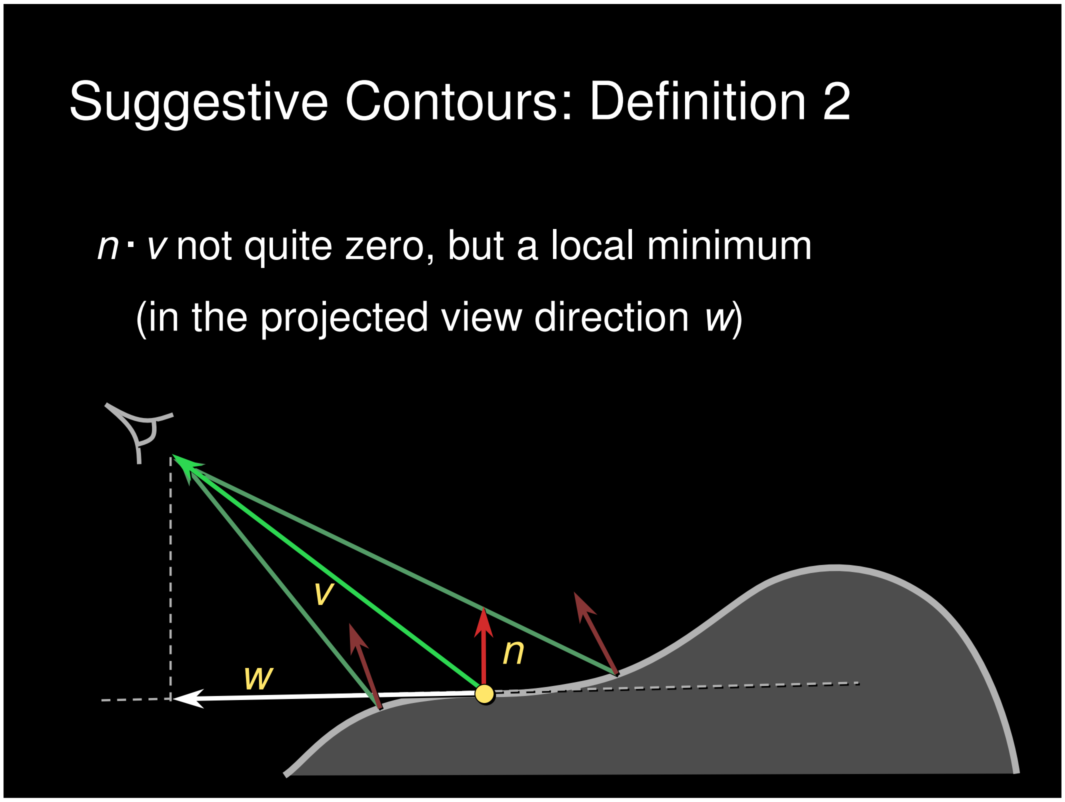
- Start with per-vertex normals
- Interpolate per-face (same as Phong shading)
- Compute n · v at each point
- Find zero crossing within face



So, there's a frequently-used alternative for polygonal meshes that tends to produce much nicer curves. It is a bit similar to Phong shading, in that it involves starting with per-vertex normals and interpolating them across a face. Once you know n at each point, you can find n dot v, and locate the curve on the face that corresponds to n dot v = 0. A slightly simpler variant of this is to just compute n dot v at the vertices, interpolate across the face, and figure out where the zero crossing is.



Now that we've seen contours, let's move on to defining suggestive contours. Here's the first definition: contours in nearby views. What happens if we start with the viewpoint at top (which produces a contour), then move the viewpoint down a little bit? First, the contour slides along a surface to a new location where its surface normal is perpendicular to the new view direction. But something else happens here too. A new contour appears that does not correspond to any in a closer viewpoint. This is a suggestive contour from the original viewpoint. The other contour corresponds to that in the original viewpoint, and is not a suggestive contour. Adding this qualification to our definition completes it.



While intuitive, the first definition doesn’t really lead to efficient algorithms for computing suggestive contours. So, let’s look at a second definition (which can be proven equivalent to the first one). The idea is that suggestive contours are places where n dot v doesn’t quite make it to zero (at which point we’d have a contour), but is a local minimum on the surface. That is, the location of a suggestive contour from this viewpoint (assumed to be distant) is where the normal is locally closest to perpendicular to the view direction, as you consider points along this normal slice of the surface. This involves moving in the direction “w”, which we define to be the projection of v, the view direction, into the local tangent plane of the surface.

Minima vs. Zero Crossings

Definition 2: Minima of $n \cdot v$

Finding minima is equivalent to:

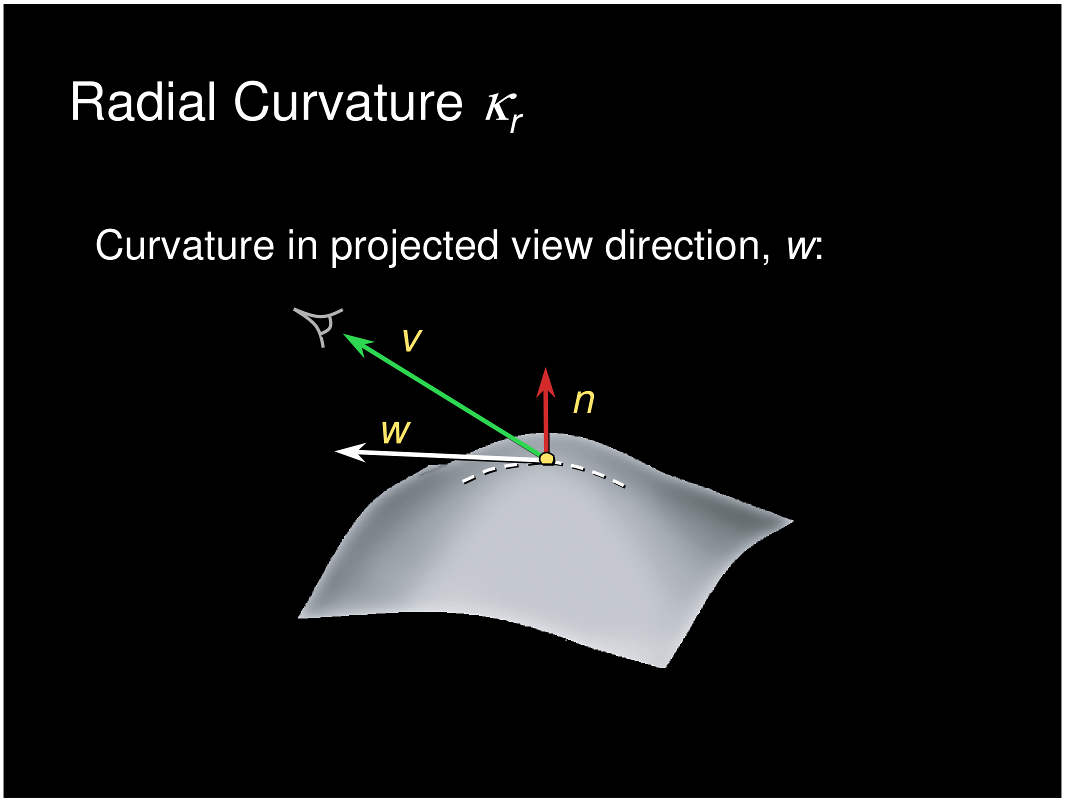
finding zeros of the derivative

checking that 2nd derivative is positive

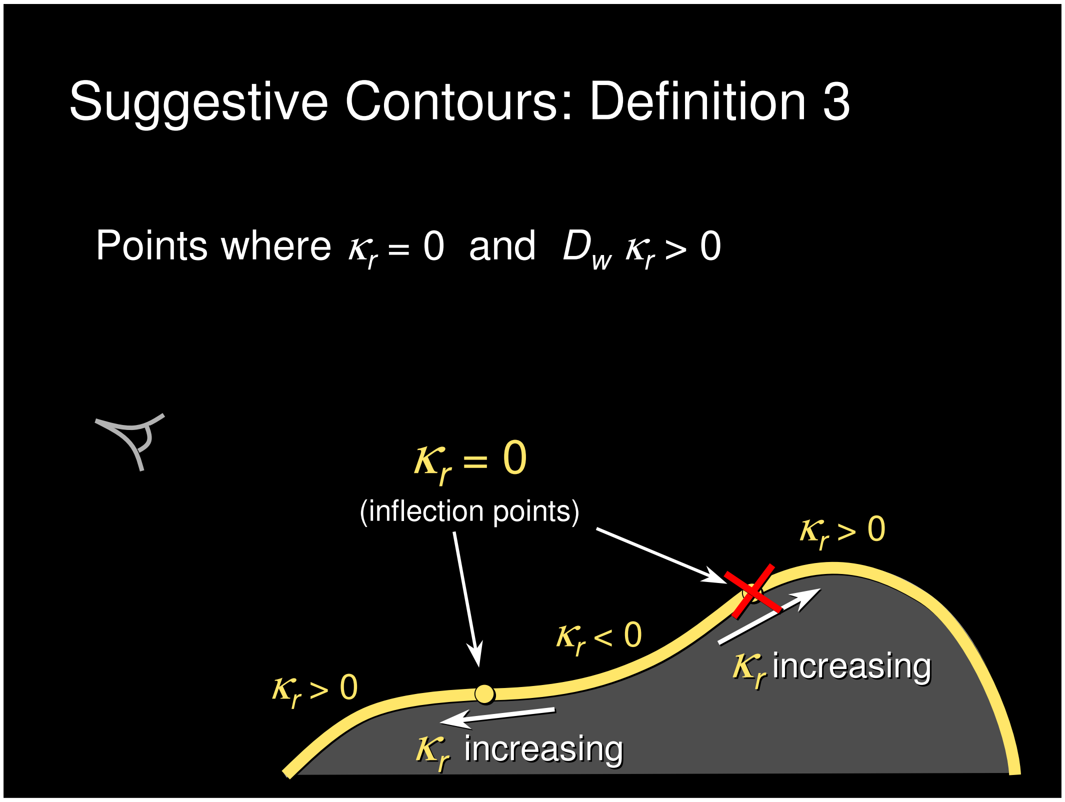
This leads to definition 3.

Derivative of $n \cdot v$ is a form of curvature…

While definition 2 is better from the point of view of computation, we can transform it into yet another form that is still more convenient. The basic idea is that we're looking for local minima, so we use the usual definition that minima are places where the derivative is zero, and the next higher-order derivative is positive (which distinguishes them from maxima). Now, it turns out that derivatives of n dot v are related to curvature.



In particular, the derivative of n dot v has the same zeros as a quantity called "radial curvature", which is just the curvature in the direction w (which, you'll recall, is the projection of the view direction).



So, our third definition of suggestive contours is that they are zeros of radial curvature, subject to a derivative test. This test needs to enforce that the "directional derivative" of radial curvature, in the direction w, is positive. To figure out what that is, we'll need to go back and look at the next higher order of surface differentials.

Derivative of Curvature

Just as $\mathbf{II} = \begin{pmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{pmatrix}$, can define $\mathbf{C} = \begin{pmatrix} \frac{\partial \mathbf{II}}{\partial u} & \frac{\partial \mathbf{II}}{\partial v} \end{pmatrix}$

C is a rank-3 tensor or "cube of numbers"

Symmetric, so 4 unique entries: $\mathbf{C} = \begin{bmatrix} P^{Q} & Q^{S} \\ Q^{S} & S^{T} \end{bmatrix}$

Multiplying by a direction three times gives (scalar) derivative of curvature

Once we know about curvatures, derivatives of curvature are really nothing special. The only really interesting thing is that, as opposed to the normal (which was a vector) and the second fundamental matrix II, the derivative of curvature is now a three-dimensional tensor, which can be thought of as a vector of matrices or as a "cube of numbers". In order to get the derivative of curvature in a particular direction, you multiply this tensor by that direction three times.

Finding Suggestive Contours

Finding κ_r:

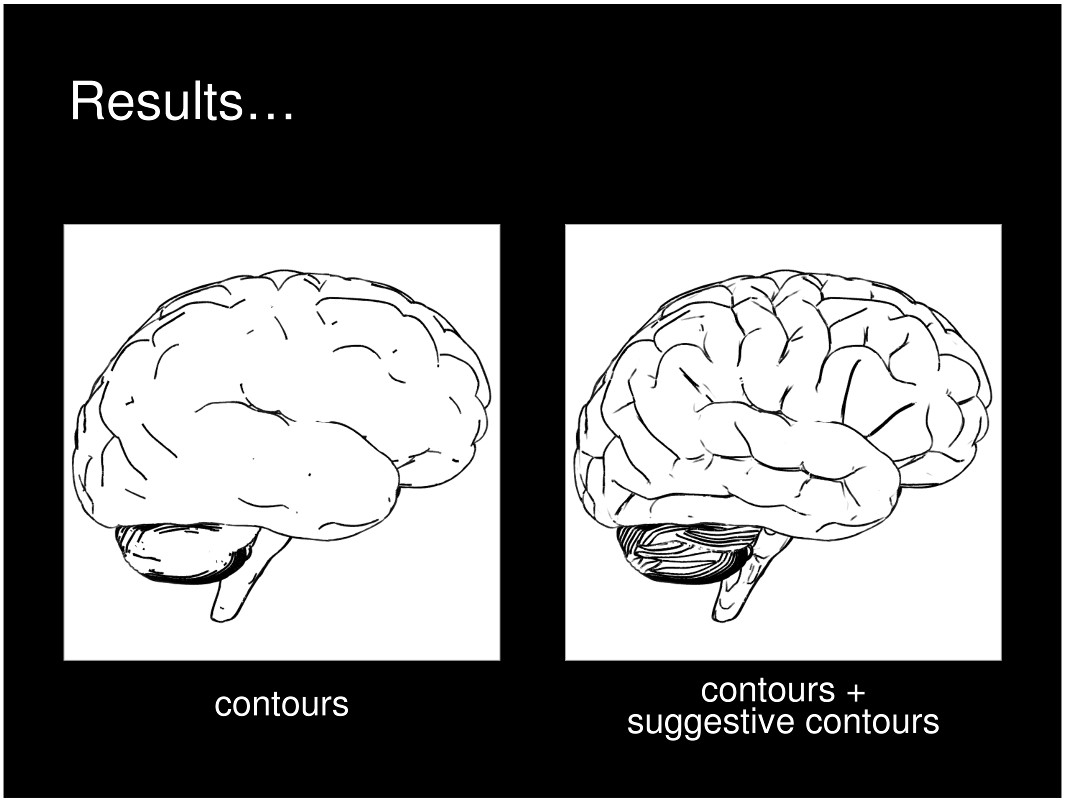
$$\kappa_r = \mathbf{II}(\hat{w}, \hat{w})$$

Finding $D_w\kappa_r$: (extra term due to chain rule)

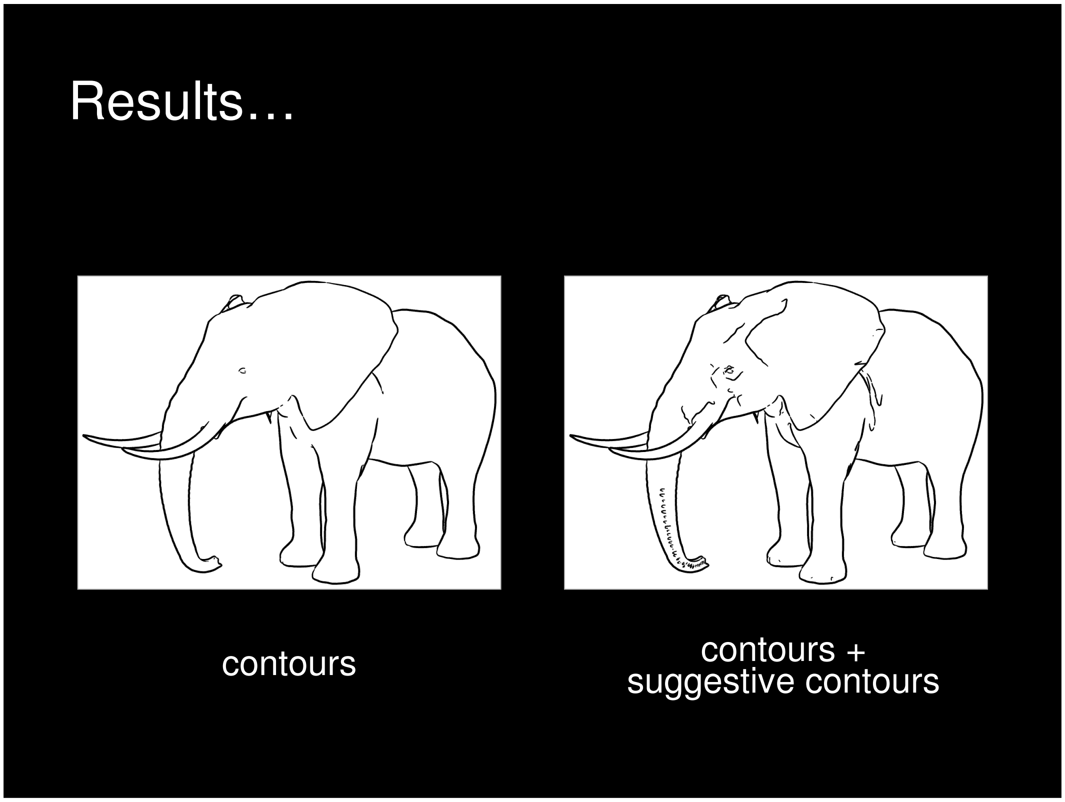
$$D_{\hat{w}}\kappa_r = \mathbf{C}(\hat{w}, \hat{w}, \hat{w}) + 2K\cot\theta,$$

$$\text{where } \kappa_r = 0$$

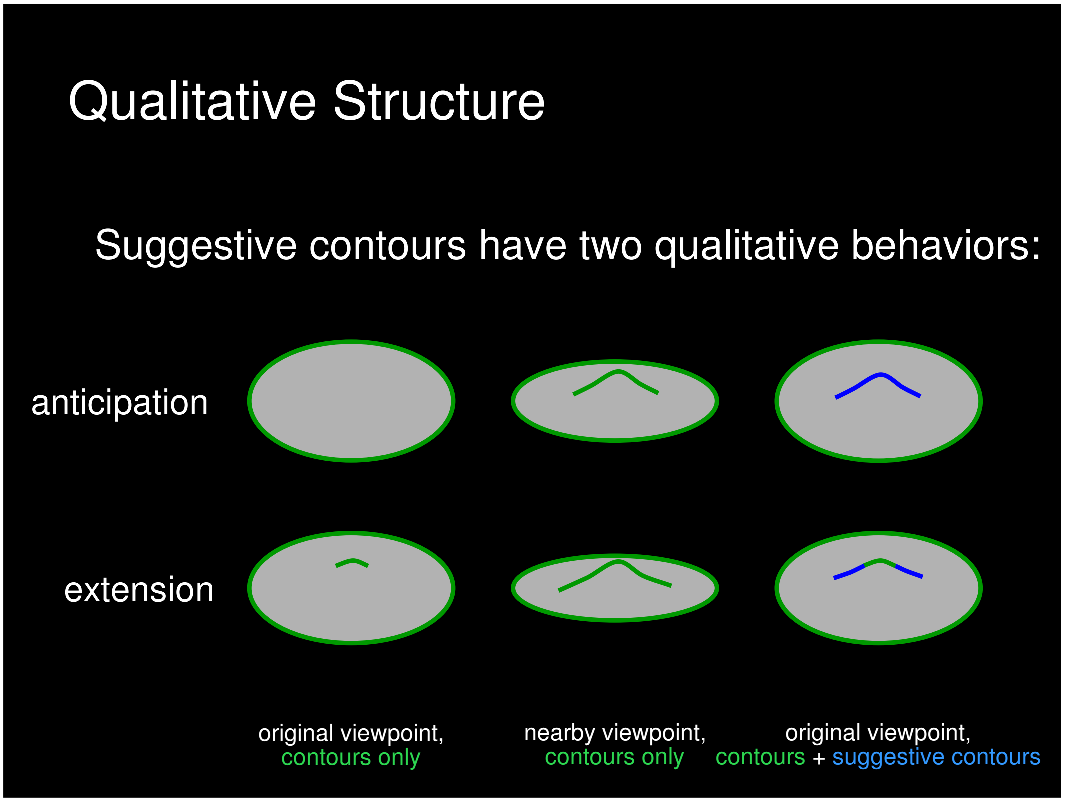
So, to recap, the most computationally convenient definition of suggestive contours involves finding the zeros of radial curvature, which you compute by multiplying II by w twice, then checking the sign of the directional derivative of radial curvature, which you get by multiplying the C tensor by w three times (there's also an extra term due to the chain rule, which accounts for the change of w itself as you move in the w direction).



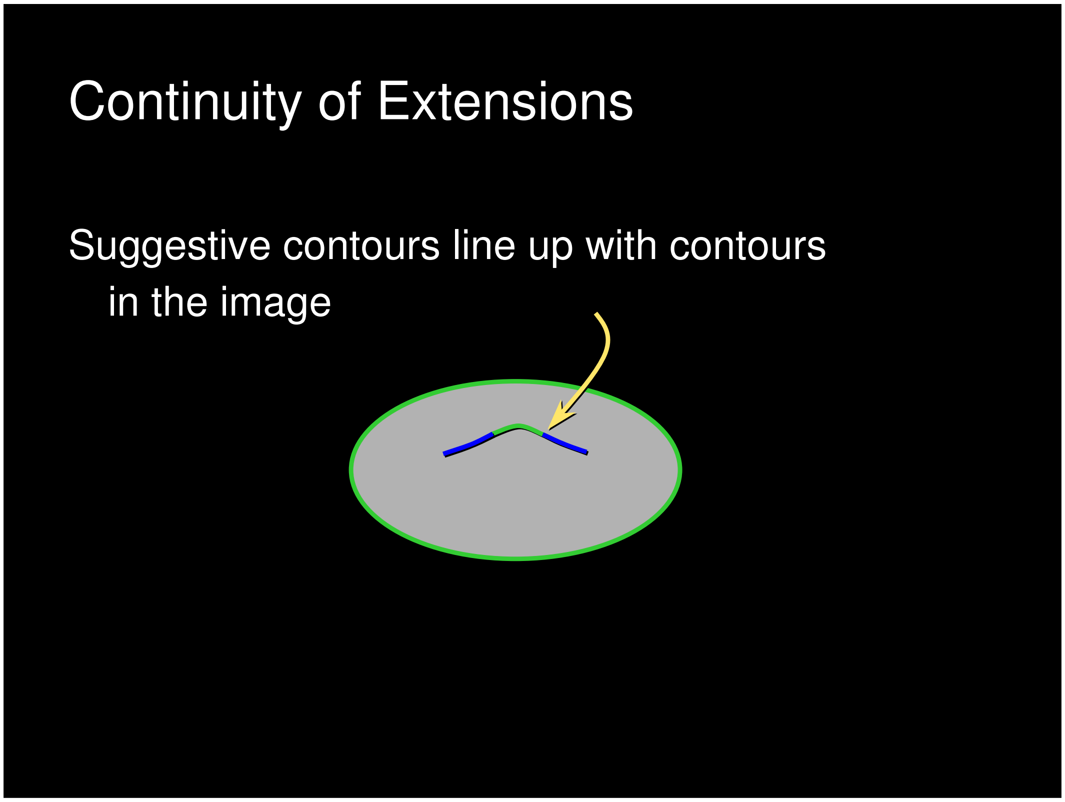
Here is your brain on contours. Here is your brain on suggestive contours. Any questions?



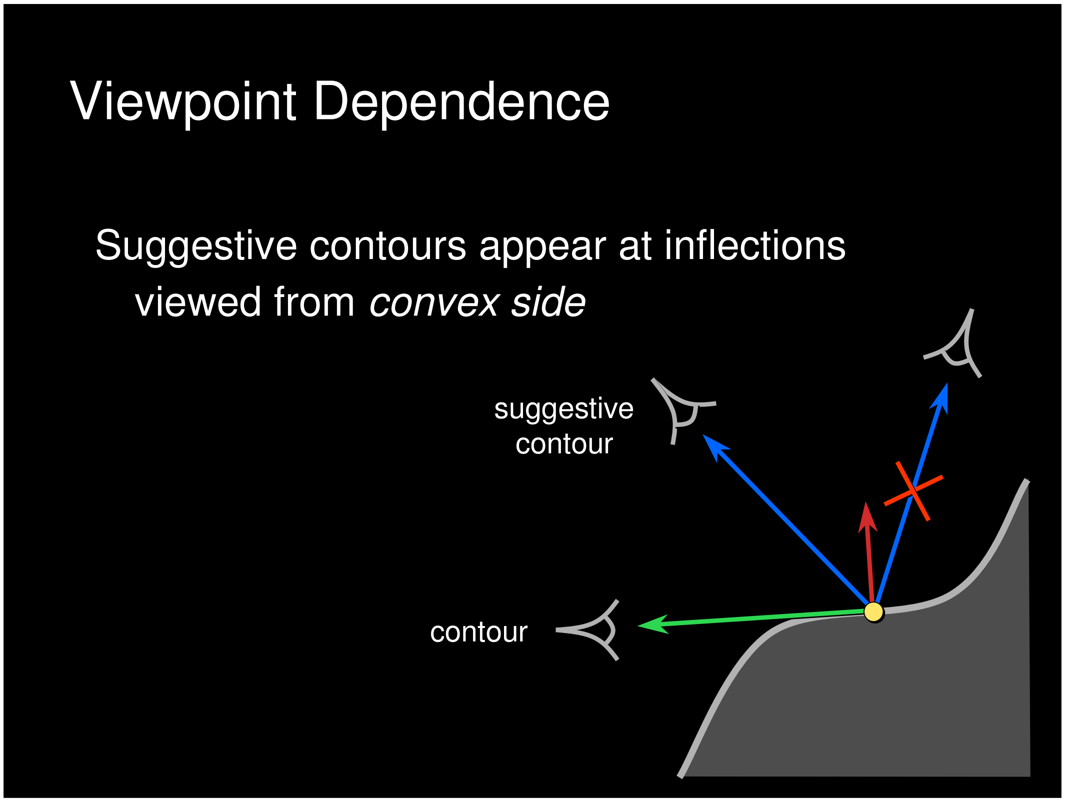
Here is your elephant on contours… (rest of not-a-joke omitted in the interests of good taste)



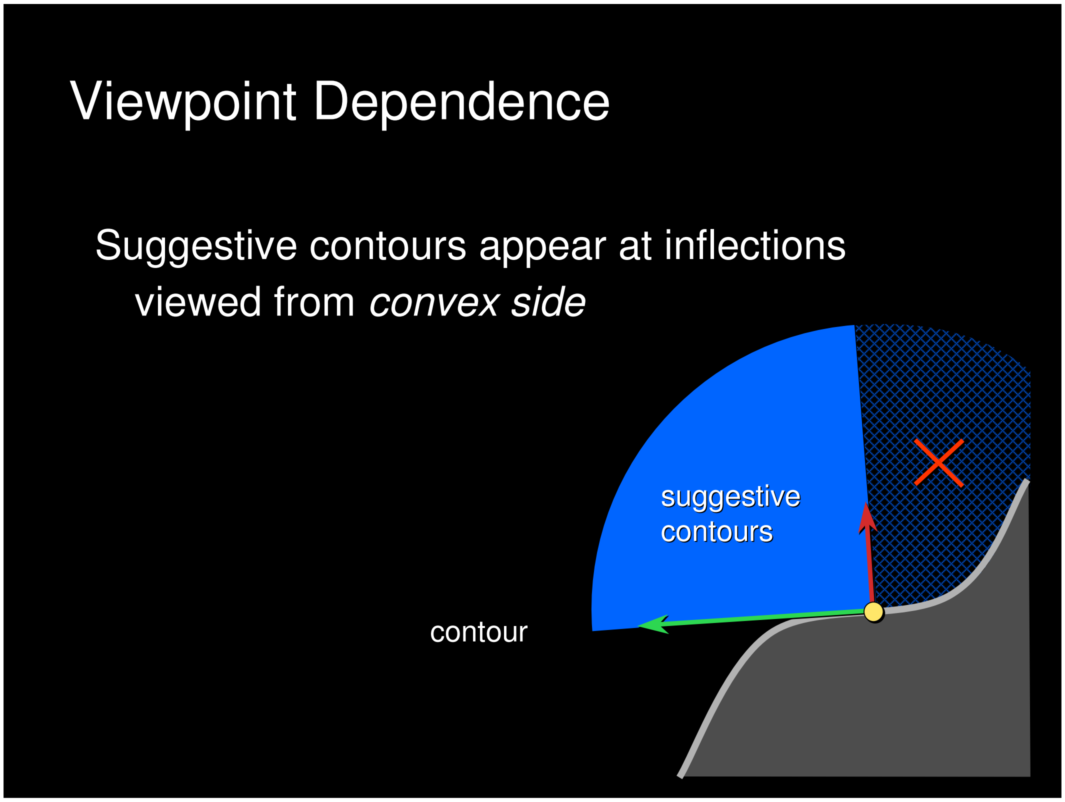
It turns out that suggestive contours have some nice properties that let them complement contours very nicely. First, they can either "anticipate" contours by showing up in nearby viewpoints (i.e., definition 1), or "extend" contours in one view. Here we use the color convention that contours are green while suggestive contours are drawn in blue.



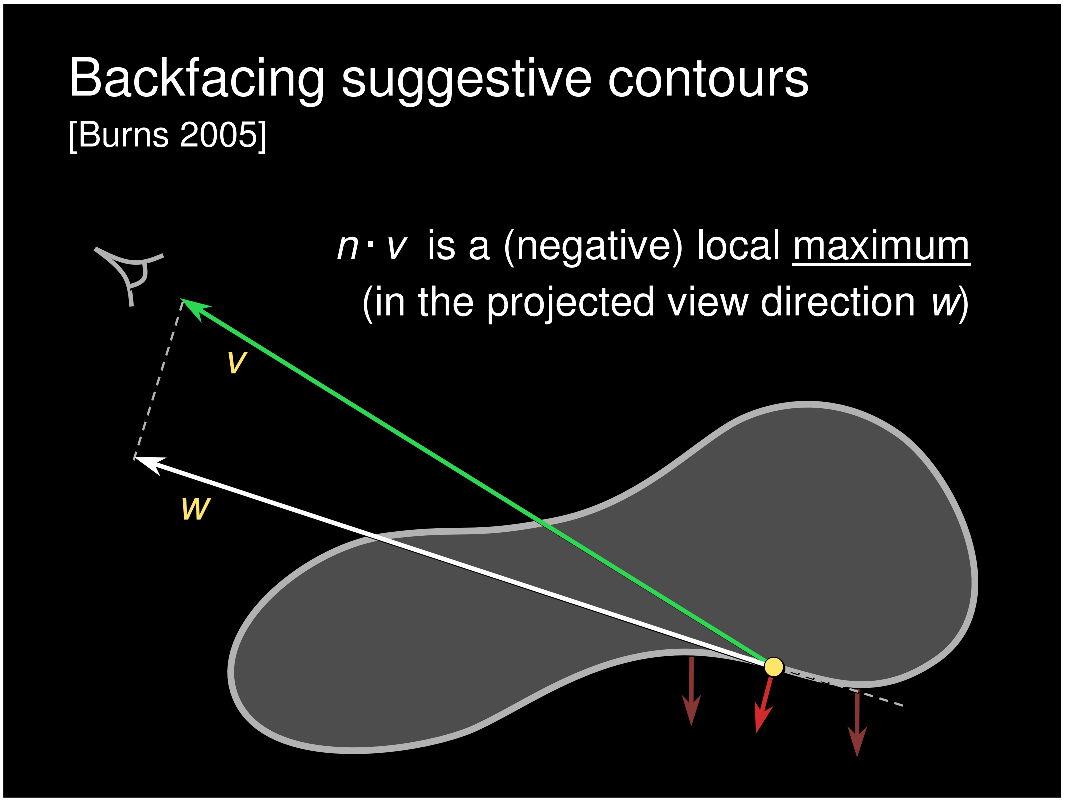
Moreover, in the case of extension, the suggestive contours line up (with G1 continuity) with contours in the image.



Another property that becomes apparent from definition 3 is that contours show up at inflections (of normal curves) on the surface, but only when viewed from the convex side. In this case, the derivative test eliminates the suggestive contour at the rightmost viewpoint.



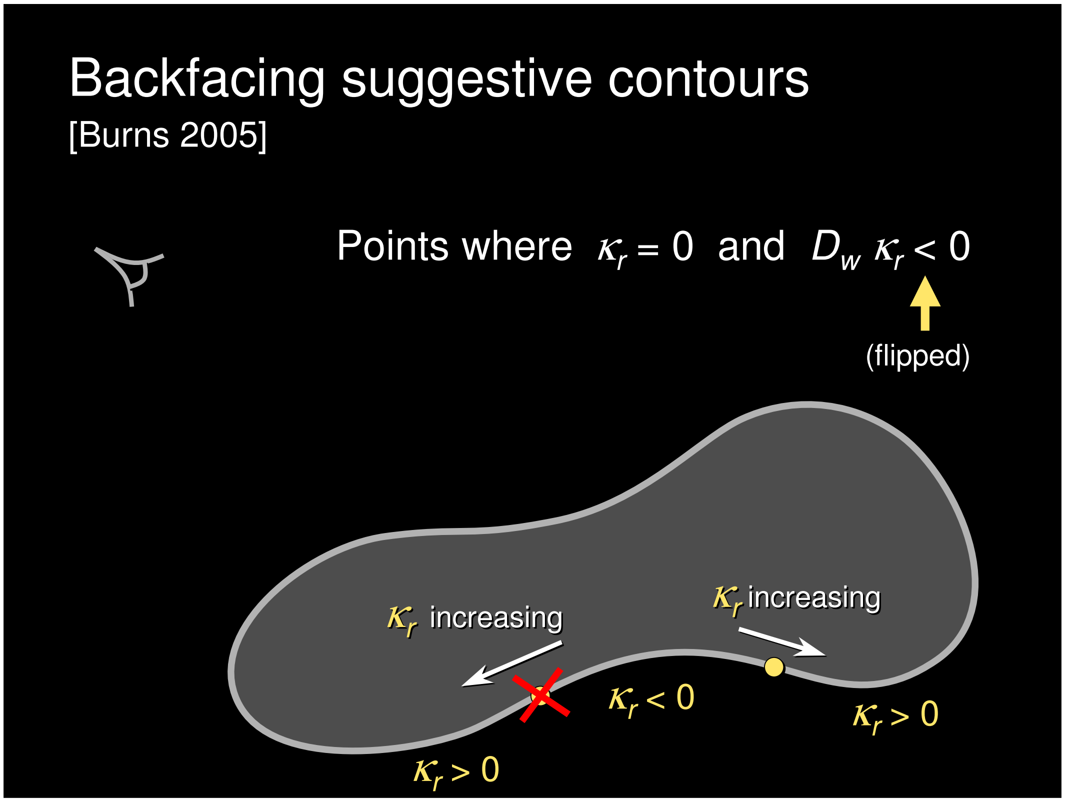
So, considering an inflection on a surface, there is some region of viewpoints from which suggestive contours get drawn, a region where they don't, and a threshold direction at which you start getting contours.



We also need to consider what happens with suggestive contours on backfacing parts of the object; such lines are an effective ingredient in transparent renderings. In fact, we do just this in our paper on volumetric line drawings here at Siggraph this year.

The first definition of suggestive contours still applies: contours in nearby viewpoints.

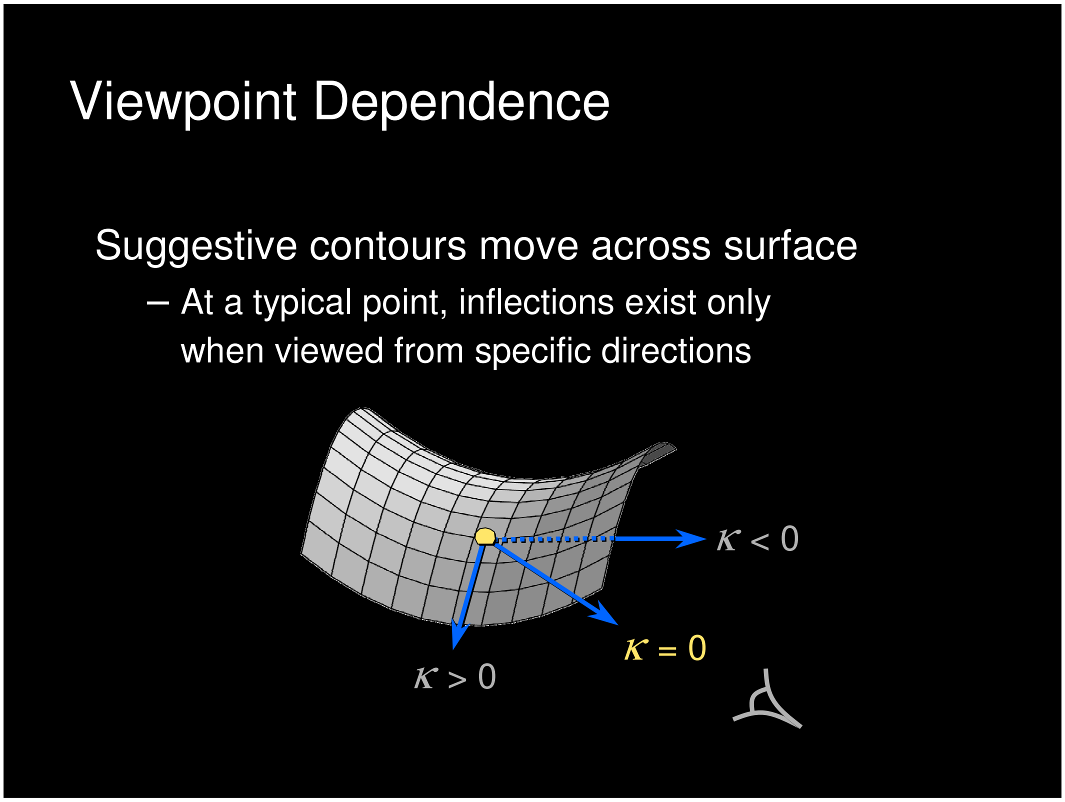
We need to change definition 2 a little bit. When looking at how values of n dot v change across the surface, we are still looking for places where n dot v is almost but doesn't quite reach zero. The difference is that we're now looking for *maxima* of n dot v: negative maxima.



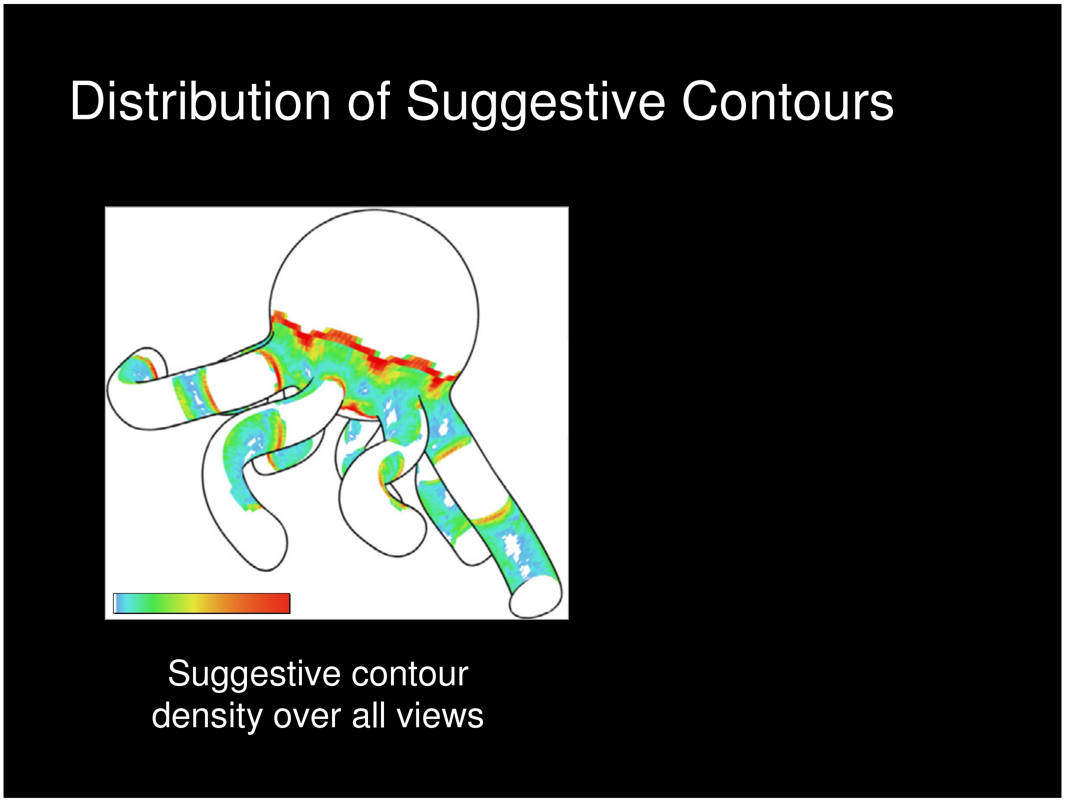
For the third definition, we are now looking for places where the radial curvature is zero where the radial curvature is increasing AWAY from the camera. So the sign of our derivative test gets flipped.

Perhaps a good way of thinking about this is that for backfaces, we're just considering what the inside-out version of the surface looks like (where the normals and curvatures are negated).

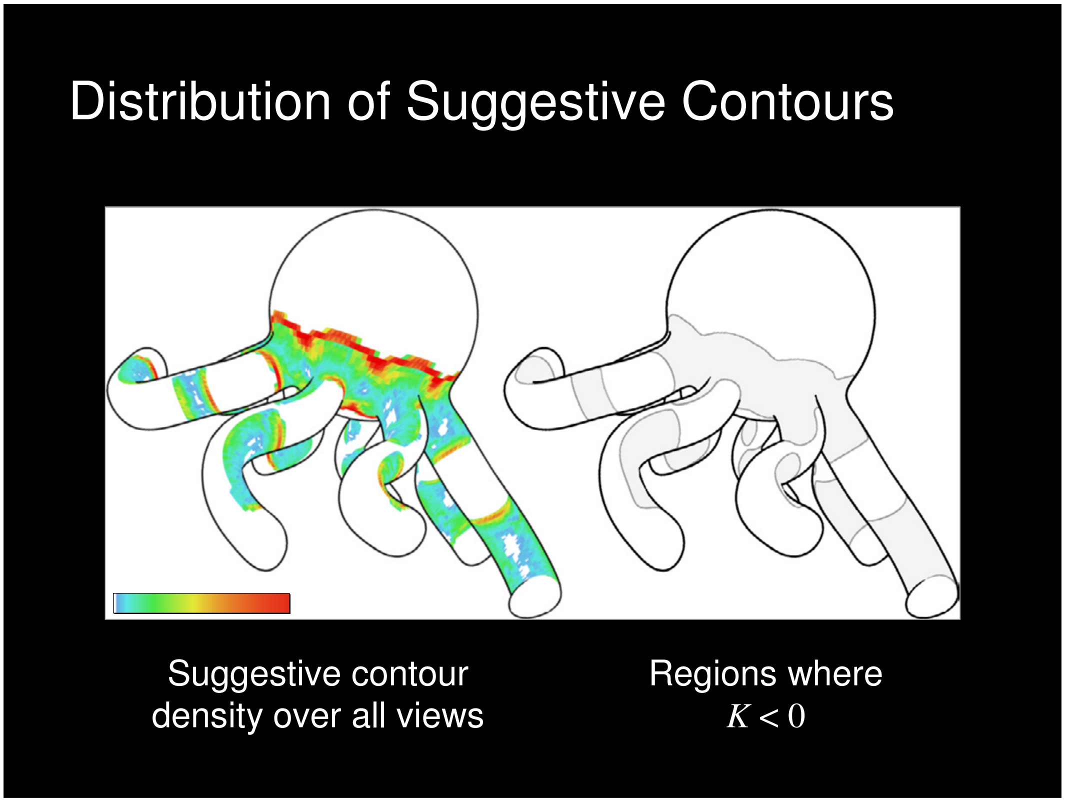
Of course, these suggestive contours still smoothly extend transparently rendered contours.



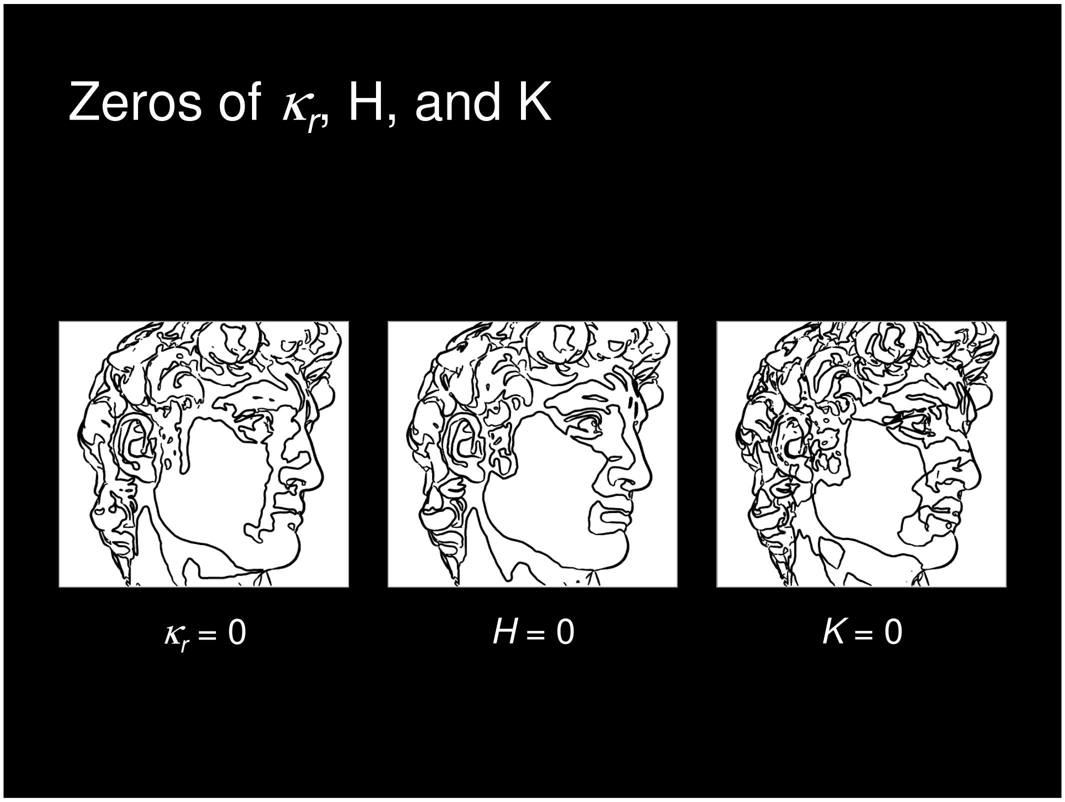
Moving the viewpoint out of the plane, we see that suggestive contours can only happen when a surface is viewed from a very particular direction such that the curvature is zero. If you rotated the viewpoint one way, you'd get positive curvatures, and negative curvatures if you rotated the other way. Note also that having a direction for which the curvature is zero implies that the principal curvatures (which are the minimum and maximum limits for normal curvature) can't be both positive or both negative. This, in turn, implies that in order to get a suggestive contour, the Gaussian curvature must be negative (or at worst zero).



This property can be illustrated empirically as well. Here we've taken a model and plotted a histogram of how many views have suggestive contours at each point on the surface.

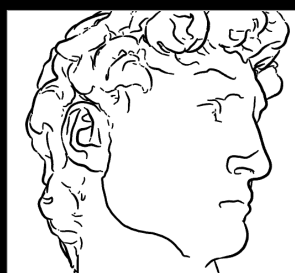


Comparing this to regions of negative Gaussian curvature, we see complete agreement. We also see the surprising fact that suggestive contours tend to hug the lines of zero Gaussian curvature (i.e., our friends the parabolic lines). We'll see later how to show this mathematically, but meanwhile let's think back to Klein's experiment. Even if the suggestive contours were always close to the parabolic lines, there's still a big difference between drawing them and our definition of suggestive contours: the derivative test.

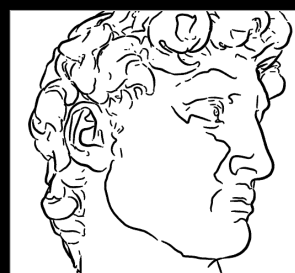


In fact, if we didn't apply the derivative test, the lines of zero radial curvature would look pretty bad: just as bad as drawing all the parabolic lines. (Here we also show zeros of mean curvature for the sake of completeness.)

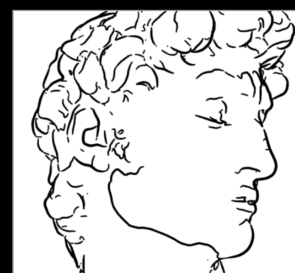
Zeros of κ_r, H, and K (with derivative tests)



$\kappa_r = 0$
$D_w \kappa_r > 0$



$H = 0$
$D_w H > 0$



$K = 0$
$D_w K > 0$

If we add a derivative test, you can see that parabolic lines suddenly don’t look so bad, though in general the suggestive contours still look better (and have the other properties of lining up with contours, etc.)

Lines and Surface Differentials

Can classify the lines we've seen according to:

- Derivative order (normal = 1st, curvature = 2nd, etc.)
- View independent / dependent

So, to sum up, the most useful lines in NPR tend to involve surface differentials. It's interesting to form a catalog of them based on the order of derivatives involved, as well as whether they are view-dependent or view-independent.

It's important to note that there's a lot more to drawing lines than just these basic definitions: a practical system will filter out some subset of these lines, figure out which ones are visible, and apply some sort of smoothing or stylization.

Lines and Surface Differentials

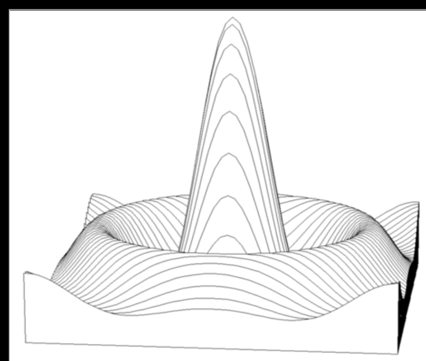
Zeroth-order, view independent: elevation lines

[USGS]

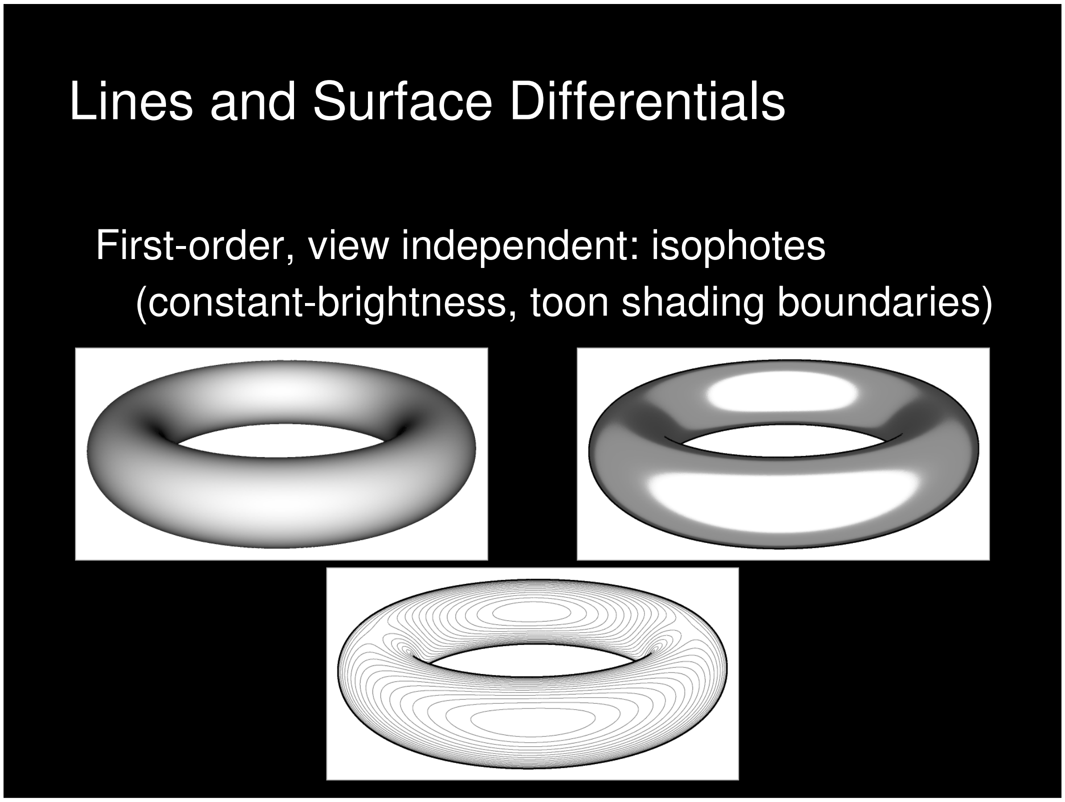
Starting off, we have lines formed just from the surface positions (the 0-th order derivative). We really haven't looked at these, but they are fairly straightforward. One flavor is the constant-altitude lines found on topographic maps.

Lines and Surface Differentials

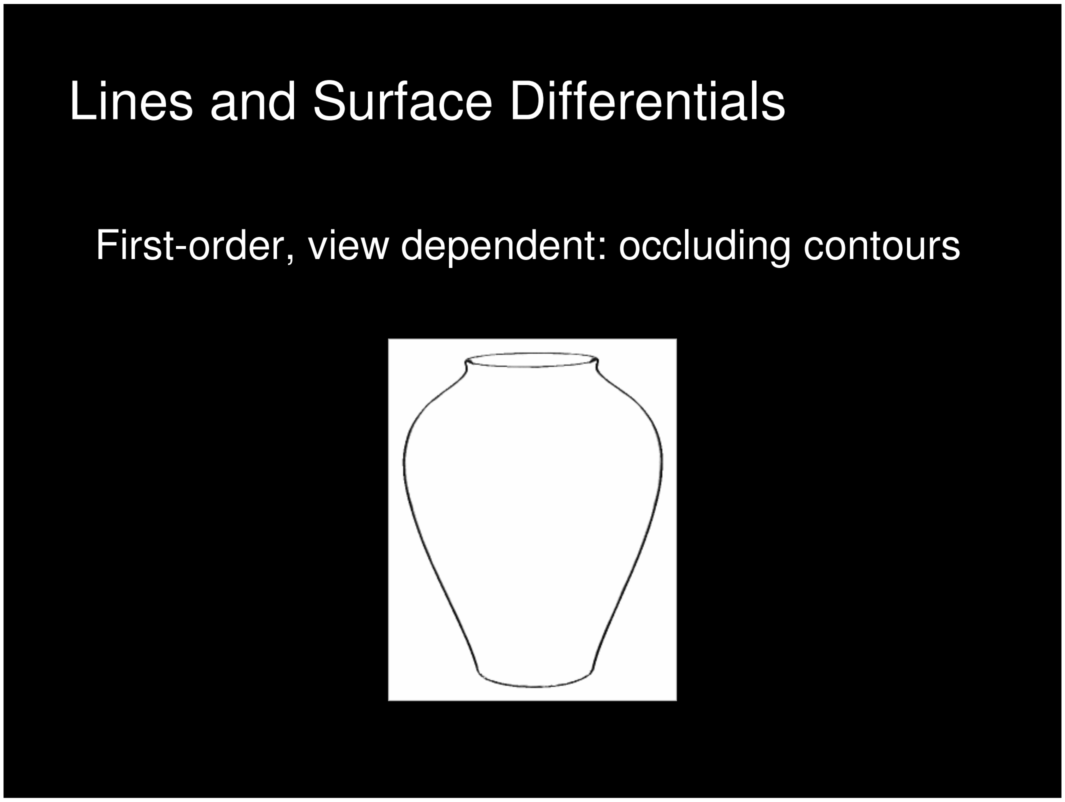
Zeroth-order, view dependent: cutting lines



Here’s another example of similar lines, where now the surface has been “sliced” by planes perpendicular to the view direction.



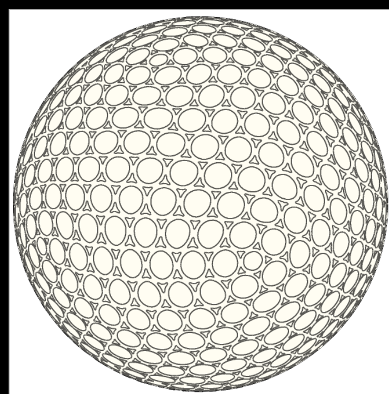
Moving to the first order differentials, or normals, you can draw lines based on isovalues of n dot the light direction, which are called isophotes. These are also the boundaries between toon shading regions.



Of course, occluding contours and silhouettes are the most ubiquitous example of first-order, view-dependent lines.

Lines and Surface Differentials

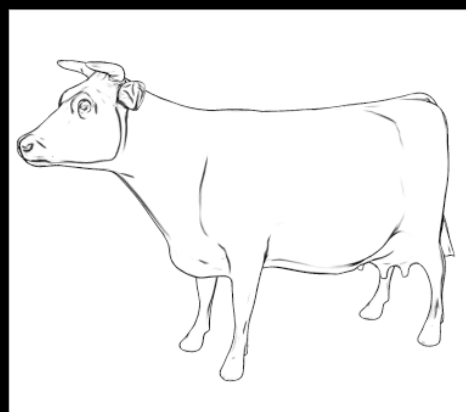
Second-order, view independent: parabolic lines



Parabolic lines are a good example of second-order view-independent lines…

Lines and Surface Differentials

Second-order, view dependent: suggestive contours

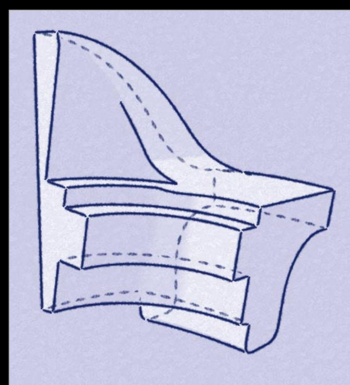


… while suggestive contours are view dependent.

Lines and Surface Differentials

Third-order, view independent: ridges and valleys

- Local minima/maxima of curvature, in minimum/maximum principal directions



Finally, ridges and valleys involve local extrema of curvature, hence are really third-order.