Notes on Differential Geometry

Defining and extracting suggestive contours, ridges, and valleys on a surface requires an understanding of the basics of differential geometry. Here we collect the necessary background, specialized to the case of surfaces in 3D and considering only orthonormal bases. For more details, consult [Cipolla and Giblin 2000, do Carmo 1976, Koenderink 1990].

We begin by considering a smooth and closed surface S and a point $\mathbf{p} \in S$ sitting on the surface.

First-order structure: The first-order approximation of this surface around this point is the tangent plane there; the unit normal vector $\mathbf{n}$ at $\mathbf{p}$ is perpendicular to this plane. (We use outward-pointing normal vectors.) Directions in the tangent plane at $\mathbf{p}$ can be described with respect to three-dimensional basis vectors $\{\mathbf{s}_u, \mathbf{s}_v\}$ that span the tangent plane. Generally, the three-dimensional vector $\mathbf{x}$ that sits in the tangent plane is written in local coordinates as $[u\ v]^{\mathrm{T}}$, where $\mathbf{x} = u\,\mathbf{s}_u + v\,\mathbf{s}_v$.

Second-order structure: The unit normal $\mathbf{n}$ is a first-order quantity; it turns out that the interesting second-order structures involve derivatives of normal vectors. A *directional derivative* of a function defined on the surface (the unit normal being one possible function) specifies how that function changes as you move in a particular tangent direction. For instance, the directional derivative $D_{\mathbf{x}}\mathbf{n}$ at $\mathbf{p}$ characterizes how $\mathbf{n}$ "tips" as you move along the surface from $\mathbf{p}$ in the direction $\mathbf{x}$. (This same relationship can be conveyed by the differential $d\mathbf{n}(\mathbf{x})$, which describes how $\mathbf{n}$ changes as a function of a particular tangent vector $\mathbf{x}$.) Since derivatives of unit vectors must be in perpendicular directions, the derivatives of $\mathbf{n}$ lie in the tangent plane. This directional derivative can be written as:

$$D_{\mathbf{x}}\mathbf{n} = n_u\,\mathbf{s}_u + n_v\,\mathbf{s}_v, \tag{1}$$

where

$$\begin{bmatrix} n_u \\ n_v \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2}$$

where the entries L, M and N depend on the local surface geometry—see [Cipolla and Giblin 2000] for details. Note that $D_{\mathbf{x}}\mathbf{n}$ depends linearly on the length of $\mathbf{x}$.

Now, suppose we have two tangent vectors $\mathbf{x}_1 = u_1\,\mathbf{s}_u + v_1\,\mathbf{s}_v$ and $\mathbf{x}_2 = u_2\,\mathbf{s}_u + v_2\,\mathbf{s}_v$. The *second fundamental form* $\mathbf{II}$ at $\mathbf{p}$ is a symmetric bilinear form specified by:

$$\begin{aligned} \mathbf{II}(\mathbf{x}_1, \mathbf{x}_2) &= (D_{\mathbf{x}_1}\mathbf{n}) \cdot \mathbf{x}_2 = (D_{\mathbf{x}_2}\mathbf{n}) \cdot \mathbf{x}_1 \\ &= \begin{bmatrix} u_1 & v_1 \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}. \end{aligned} \tag{3}$$

(This differs in sign from [do Carmo 1976] due to our choice of *outward* pointing normals.) Since $\mathbf{II}$ is symmetric, we use $\mathbf{II}(\mathbf{x})$ as a shorthand to indicate only one vector product has been performed; so $\mathbf{II}(\mathbf{x}) = D_{\mathbf{x}}\mathbf{n}$, and $\mathbf{II}(\mathbf{x}_1,\mathbf{x}_2) = \mathbf{II}(\mathbf{x}_1)\cdot\mathbf{x}_2 = \mathbf{II}(\mathbf{x}_2)\cdot\mathbf{x}_1$.

The *normal curvature* of a surface S at a point $\mathbf{p}$ measures its curvature in a specific direction $\mathbf{x}$ in the tangent plane, and is defined in terms of the second fundamental form. The normal curvature, written as $\kappa_n(\mathbf{x})$, is:

$$\kappa_n(\mathbf{x}) = \frac{\mathbf{II}(\mathbf{x},\mathbf{x})}{\mathbf{x}\cdot\mathbf{x}}.$$

Notice how the length and sign of $\mathbf{x}$ do not affect the normal curvature. On a smooth surface, the normal curvature varies smoothly with direction $\mathbf{x}$, and ranges between the principal curvatures κ_1 and κ_2 at $\mathbf{p}$. These are realized in their respective principal curvature directions $\mathbf{e}_1$ and $\mathbf{e}_2$, which are perpendicular and unit length.

The Gaussian curvature K is equal to the product of the principal curvatures: $K = \kappa_1\kappa_2$, and the mean curvature H is their average: $H = (\kappa_1+\kappa_2)/2$. Wherever K is strictly negative (so that only one of κ_1 or κ_2 is negative), there are two directions along which the curvature is zero. These directions, called the *asymptotic directions*, play a central role in where suggestive contours are located.

The vector $D_{\mathbf{x}}\mathbf{n}$ can be broken into two components; in the direction of $\mathbf{x}$, and perpendicular to it. The length of the component in the direction of $\mathbf{x}$ is simply the normal curvature κ_n. The length of the perpendicular component is known as the *geodesic torsion* τ_g, and describes how much the normal vector tilts to the side as you move in the direction of $\mathbf{x}$. If we define the perpendicular direction $\mathbf{x}_\perp$ as:

$$\mathbf{x}_\perp = \mathbf{n}\times\mathbf{x}$$

then we have:

$$D_{\mathbf{x}}\mathbf{n} = \mathbf{II}(\mathbf{x}) = \kappa_n(\mathbf{x})\,\mathbf{x} + \tau_g(\mathbf{x})\,\mathbf{x}_\perp \tag{4}$$

It follows that $\tau_g(\mathbf{x}) = \mathbf{II}(\mathbf{x},\mathbf{x}_\perp)/\mathbf{x}\cdot\mathbf{x}$. Furthermore, when $\mathbf{x}$ is an asymptotic direction, $D_{\mathbf{x}}\mathbf{n}$ is perpendicular to $\mathbf{x}$ and it can be shown that $\tau_g^2(\mathbf{x}) = -K$ [Koenderink 1990].

Principal coordinates: Using the principal directions $\{\mathbf{e}_1,\mathbf{e}_2\}$ as the local basis leads to *principal coordinates*. In principal coordinates, the matrix in equations (2) and (3) is diagonal with the principal curvatures as entries:

$$\begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}.$$

Given $[u\ v]^{\mathrm{T}} = [\cos\phi\ \sin\phi]^{\mathrm{T}}$, where ϕ is the angle in the tangent plane measured between a particular direction and $\mathbf{e}_1$, this leads to the well-known Euler formula for normal curvature:

$$\kappa_n(\phi) = \kappa_1\cos^2\phi + \kappa_2\sin^2\phi$$

and the following for the geodesic torsion:

$$\tau_g(\phi) = (\kappa_2-\kappa_1)\sin\phi\cos\phi$$

(where the sign of τ_g depends on our definition of $\mathbf{x}_\perp$, above).

Third-order structure: In order to define ridges and valleys, and to analyze how suggestive contours move across a surface, we will need additional notation that describes *derivatives of curvature*. The gradient $\nabla \kappa_r$ is a vector in the tangent plane that locally specifies the magnitude and direction of maximal change in κ_r on the surface.

In the following, we use principal coordinates, as the third-order derivatives are much simpler to state. In this case, finding derivatives of normal curvatures involves taking the directional derivative of $\mathbf{II}$ in a particular tangent direction $\mathbf{x}$. The result is written in terms of a symmetric trilinear form $\mathbf{C}$, built from a $2 \times 2 \times 2$ (rank-3) tensor whose entries depend on the third derivatives of the surface [Gravesen and Ungstrup 2002]. Such derivatives have been ingredients in measures of fairness for variational surface modeling [Gravesen and Ungstrup 2002, Moreton and Séquin 1992].

We write $\mathbf{C}$ with either two or three arguments—indicating how many times a vector is multiplied onto the underlying tensor. Thus, $\mathbf{C}(\mathbf{x},\mathbf{x})$ is a vector and $\mathbf{C}(\mathbf{x},\mathbf{x},\mathbf{x})$ is a scalar. The order of the arguments does not matter, as $\mathbf{C}$ is symmetric. In principal coordinates, the tensor describing $\mathbf{C}$ has 4 unique entries [Gravesen and Ungstrup 2002]:

$$P = D_{\mathbf{e}_1}\kappa_1,\ Q = D_{\mathbf{e}_2}\kappa_1,\ S = D_{\mathbf{e}_1}\kappa_2,\ \text{and}\ \ T = D_{\mathbf{e}_2}\kappa_2$$

This leads to the first-order approximation of the matrix in equation (3) towards $\mathbf{x} = u\,\mathbf{e}_1 + v\,\mathbf{e}_2$ as:

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} \approx \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} + u \begin{bmatrix} P & Q \\ Q & S \end{bmatrix} + v \begin{bmatrix} Q & S \\ S & T \end{bmatrix} \tag{5}$$

(written with the tensor expanded into two matrices on the right to avoid cumbersome notation, already multiplied once by $[u\ v]^{\mathrm{T}}$.) Finally, we note how to compute the gradient and directional derivative of the normal curvature κ_n using $\mathbf{C}$:

$$\nabla \kappa_n(\mathbf{x}) = \frac{\mathbf{C}(\mathbf{x},\mathbf{x})}{\mathbf{x}\cdot\mathbf{x}} = \frac{g_u\,\mathbf{e}_1 + g_v\,\mathbf{e}_2}{\mathbf{x}\cdot\mathbf{x}},$$

where

$$\begin{bmatrix} g_u \\ g_v \end{bmatrix} = \begin{bmatrix} Pu^2 + 2Quv + Sv^2 \\ Qu^2 + 2Suv + Tv^2 \end{bmatrix}$$

and

$$\frac{D_{\mathbf{x}}\kappa_n(\mathbf{x})}{\|\mathbf{x}\|} = \frac{\mathbf{C}(\mathbf{x},\mathbf{x},\mathbf{x})}{\|\mathbf{x}\|^3} = \frac{Pu^3 + 3Qu^2v + 3Suv^2 + Tv^3}{\|\mathbf{x}\|^3}.$$

References

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